1.3 Apparent (Virtual) Forces and Coordinate Systems

Objective: Note that Newton's 2nd Law is only valid in an inertial (absolute) frame of reference. Since the Earth is a rotating, it is not an inertial (non-inertial) frame of reference.

In order to express Newton's 2nd Law in a non-inertial frame of reference (e.g., Earth), 2 additional apparent (virtual) forces are emerged from the derivations: centrifugal force and Coriolis force.

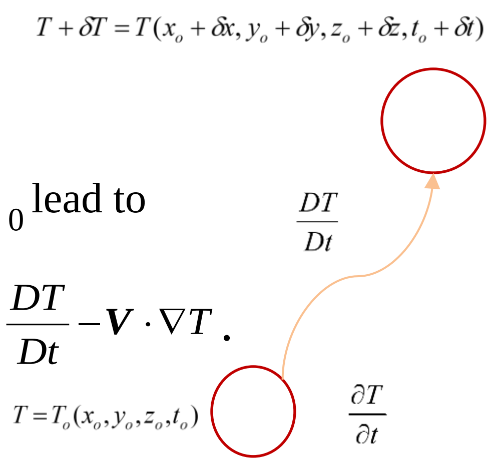
(a) Total differentiation of a scalar

Consider releasing a balloon at a place with $T = T_o(x_o, y_o, z_o, t_o)$ to another place with $T + \delta T = T(x_o + \delta x, y_o + \delta y, z_o + \delta z, t_o + \delta t)$, the temperature change following the balloon may be derived by applying the chain rule,

$$\delta T = \left(\frac{\partial T}{\partial t}\right)\delta t + \left(\frac{\partial T}{\partial x}\right)\delta x + \left(\frac{\partial T}{\partial y}\right)\delta y + \left(\frac{\partial T}{\partial z}\right)\delta z .$$

Dividing the above equation by δt and taking $\lim_{\delta T \to 0}$ lead to

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\left(\frac{\partial T}{\partial x}\right) + v\left(\frac{\partial T}{\partial y}\right) + w\left(\frac{\partial T}{\partial z}\right) \quad \text{or} \quad \frac{\partial T}{\partial t} = \frac{DT}{Dt} - \boldsymbol{V} \cdot \nabla T .$$



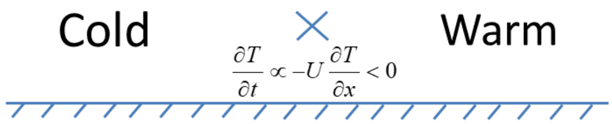
Physical meaning:

$\frac{DT}{Dt}$: Total rate of change of temperature following the motion

$\frac{\partial T}{\partial t}$: Local rate of change of temperature at a fixed location

$-V \cdot \nabla T$: Temperature advection

$$\frac{\partial T}{\partial t} = \frac{DT}{Dt} - U\frac{\partial T}{\partial x}$$



Example: Cold advection associated with a flow from cold to warm region.

(b) Total differentiation of a vector in a rotating system

For any vector $\boldsymbol{A}$, decompose it into three components in an inertial frame of reference or a rotating frame of reference (e.g., on a merry-go-around),

$$\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k} = A_x^{'}\boldsymbol{i'} + A_y^{'}\boldsymbol{j'} + A_z^{'}\boldsymbol{k'}$$

(inertial frame) (rotating frame)

Taking the total derivative of $\boldsymbol{A}$ gives

> (1) Unit vectors (***i, j, k***) do not change with time and location in an inertial frame, but (***I', j', k'***) do change in a rotating frame.
>
> (2) $\frac{D_a A_x}{Dt} = \frac{DA_x}{Dt}$ for scalars

$$\frac{D_a\boldsymbol{A}}{Dt} = \frac{DA_x}{Dt}\boldsymbol{i} + \frac{DA_y}{Dt}\boldsymbol{j} + \frac{DA_z}{Dt}\boldsymbol{k}$$

$$= \frac{DA_x^{'}}{Dt}\boldsymbol{i'} + \frac{DA_y^{'}}{Dt}\boldsymbol{j'} + \frac{DA_z^{'}}{Dt}\boldsymbol{k'} + A_x^{'}\frac{D\boldsymbol{i'}}{Dt} + A_y^{'}\frac{D\boldsymbol{j'}}{Dt} + A_z^{'}\frac{D\boldsymbol{k'}}{Dt}$$

$$= \frac{D\boldsymbol{A}}{Dt} + \left(A_x^{'}\frac{D\boldsymbol{i'}}{Dt} + A_y^{'}\frac{D\boldsymbol{j'}}{Dt} + A_z^{'}\frac{D\boldsymbol{k'}}{Dt}\right)$$

Total derivative of $\boldsymbol{A}$ in the rotating frame

The above equation leads to

$$\frac{D_a \boldsymbol{A}}{Dt} = \frac{D\boldsymbol{A}}{Dt} + \boldsymbol{\Omega} \times \boldsymbol{A}, \tag{2.2}$$

where $\boldsymbol{\Omega}$ is the rotation vector. (Example: Think about centripetal force in a circular motion.)

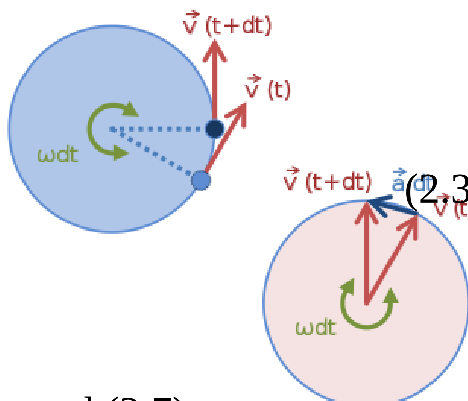
Applying (2.2) to a position vector $\boldsymbol{r}$ leads to

$$\frac{D_a \boldsymbol{r}}{Dt} = \frac{D\boldsymbol{r}}{Dt} + \boldsymbol{\Omega} \times \boldsymbol{r} \quad \text{or} \quad \boldsymbol{V}_a = \boldsymbol{V} + \boldsymbol{\Omega} \times \boldsymbol{r}\,. \tag{2.5}$$

Applying (2.2) to (2.5) again leads to

$$\frac{D_a \boldsymbol{V}_a}{Dt} = \frac{D\boldsymbol{V}}{Dt} + 2\boldsymbol{\Omega} \times \boldsymbol{V} - \Omega^2 \boldsymbol{R} \tag{2.7}$$

Note that Newton's second law states



$$\boldsymbol{a} = \frac{D_a \boldsymbol{V}_a}{Dt} = \sum_i \frac{\boldsymbol{F}_i}{m} \tag{2.3}$$

where $\boldsymbol{F}_i$'s are real forces.

Combining (2.3), real force expressions, and (2.7)
gives the equation of motion in the rotating frame of reference,

$$\frac{D_a \boldsymbol{V}_a}{Dt} = \frac{D\boldsymbol{V}}{Dt} + 2\boldsymbol{\Omega} \times \boldsymbol{V} - \Omega^2 \boldsymbol{R} = -\frac{1}{\rho}\nabla p + \boldsymbol{g}^* + \boldsymbol{F}_r$$

centripetal force

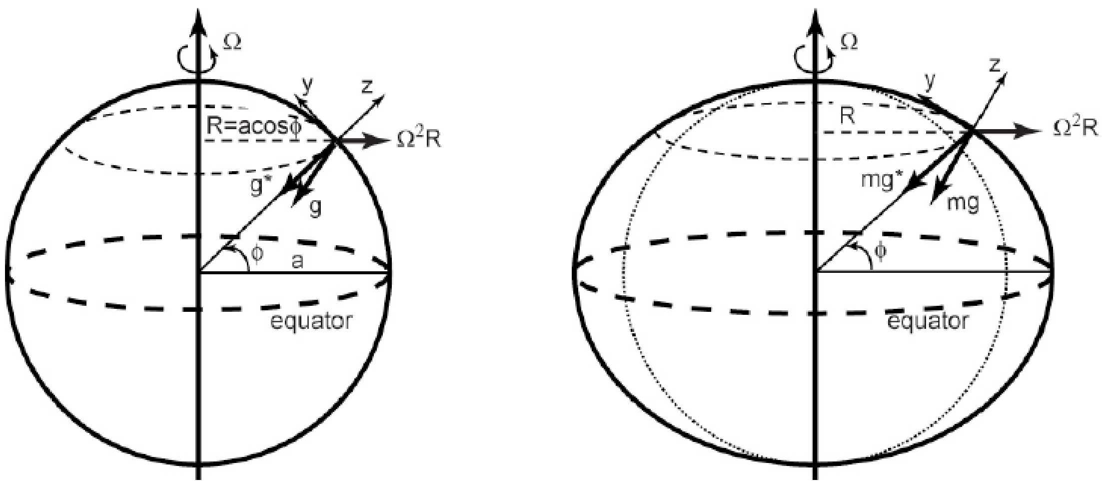
or

$$\frac{D\boldsymbol{V}}{Dt} = -\frac{1}{\rho}\nabla p - 2\boldsymbol{\Omega}\times\boldsymbol{V} + \boldsymbol{g} + \boldsymbol{F}_r \,, \tag{2.8}$$

Coriolis force

where $\boldsymbol{g} = \boldsymbol{g}^* + \Omega^2 \boldsymbol{R}$ is the effective gravity.

Video: [Visualization of the Coriolis and centrifugal forces]()

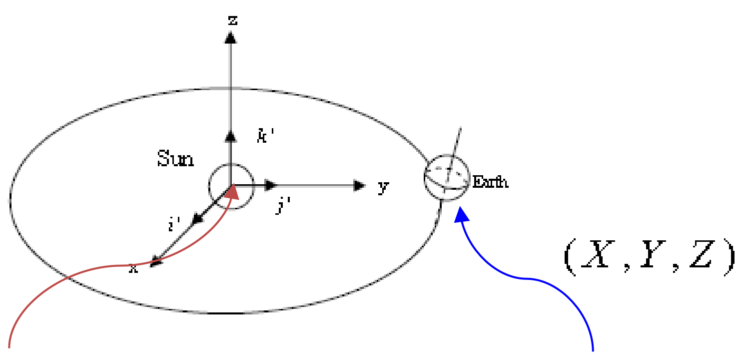


(c) Coordinate systems to be considered

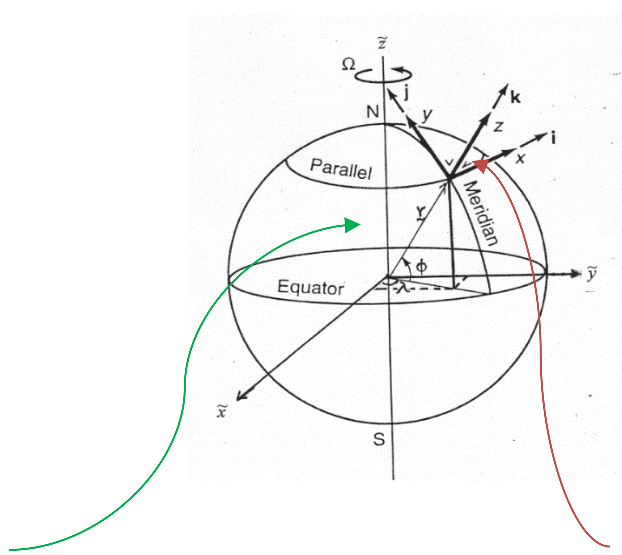
An **inertial (absolute) frame of reference** is a space-time coordinate system that neither rotates nor accelerates. In real life, such a frame of reference is purely theoretical, because gravitational force (and thus acceleration) exists everywhere in the known universe. However, they may be approximated very well in intergalactic space, or to a lesser extent within the confines of a coasting spacecraft.

- For convenience, let us assume that the solar system is an inertial frame of reference (which, in fact, is moving at a speed of ~250 km/s around the Milky Way Galaxy – see figure below) (http://www.enchantedlearning.com/subjects/astronomy/planets/earth/Speeds.shtml)

- The Earth's Rotating Frame of Reference



Inertial frame of reference (X, Y, Z) Earth rotating frame of reference



Earth spherical coordinates (λ, ϕ, r) Local coordinates (x, y, z)