ASME 434 Atmospheric Dynamics II — Dr. Yuh-Lang Lin
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Lecture Note 2 Circulation Theorems

(Holton Sec. 4.1 - Circulation Theorems; Equation editor: $D/Dt = \partial/\partial t + u\partial/\partial x$)
(2.1 Circulation Theorem, 2.2 Kelvin Circulation Theorem, 2.3 Bjierknes Lec. 2-4)

- Linear motion: measured by (linear) velocity
  Circular motion: measured by angular velocity

  $$\omega = \alpha / T$$

  There are alternative ways to measure circular motion than the angular velocity.

- Two primary measures of rotation in a fluid are:

  - Circulation – macroscopic measure for a fluid area, which is a scalar integral quantity.
  - Vorticity – microscopic measure at a point of the fluid, which is a vector.

  These quantities also allow us to apply the conservation of angular momentum to the fluid motion in an easier fashion.

➢ Definition of Circulation

The circulation, C, about a closed contour in a fluid is defined as the line integral evaluated counterclockwise along the contour of the component of the velocity vector that is locally tangent to the contour:

$$C \equiv \oint \boldsymbol{V} \cdot \boldsymbol{dl} = \oint V|dl|\cos\alpha \, .$$

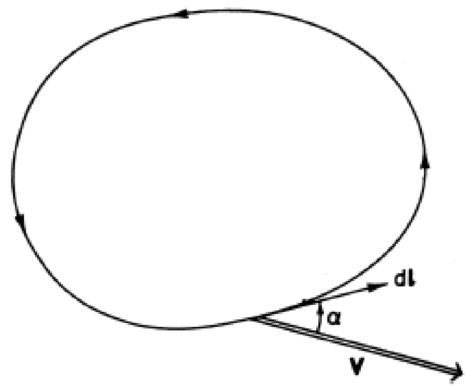


Fig. 4.1 Circulation about a closed contour.

Since

$$C = \oint \boldsymbol{V} \cdot \boldsymbol{dl} = \oint V \cos\alpha \; dl \, ,$$

$C > 0$ for cyclonic flow.

- Claim: Circulation is twice of the angular velocity times area (2Ω x Area) for a disc of fluid in a solid body rotation.

Proof: Consider a solid-body rotation.

V and ***dl*** are in the same direction all the time => $\alpha = 0$, i.e. $\cos\alpha = 1$. This gives

$$C = \oint \boldsymbol{V} \cdot \boldsymbol{dl} = \oint V dl = \int_0^{2\pi} (\Omega r)(r d\alpha) = \int_0^{2\pi} \Omega r^2 d\alpha = \Omega r^2 \int_0^{2\pi} d\alpha = 2\pi\Omega r^2 \, .$$

Thus, $$\frac{C}{\pi r^2} = 2\Omega \, .$$

That is: $\frac{Circulation}{Area} = Twice\ of\ the\ angular\ velocity$

- The circulation theorem may be derived by taking the line integral of Newton's second law for a closed chain of fluid particles, with the help of Stoke's theorem.

2.1 Circulation Theorem

Recall Eq. (2.8) in Ch. 2 (Holton's),

$$\frac{D\boldsymbol{V}}{Dt} = -2\Omega \mathrm{x} \boldsymbol{V} - \frac{1}{\rho}\nabla p - g\boldsymbol{k}, \qquad (2.8)$$

or

$$\frac{D_a \boldsymbol{V_a}}{Dt} = -\frac{1}{\rho}\nabla p - \nabla\phi, \qquad (2.8)'$$

because $\nabla\phi = \frac{\partial\phi}{\partial x}\boldsymbol{i} + \frac{\partial\phi}{\partial y}\boldsymbol{j} + \frac{\partial\phi}{\partial z}\boldsymbol{k} = \frac{\partial(gz)}{\partial z}\boldsymbol{k} = g\boldsymbol{k}$.

Take $\cdot d\boldsymbol{l}$ on both sides of (2.8)',

$$\frac{D_a \boldsymbol{V_a}}{Dt}\cdot d\boldsymbol{l} = -\frac{1}{\rho}\nabla p \cdot d\boldsymbol{l} - \nabla\phi \cdot d\boldsymbol{l}. \qquad (2.8)''$$

Left hand side can be rewritten as

$$\frac{D_a \boldsymbol{V_a}}{Dt} \cdot d\boldsymbol{l} = \frac{D}{Dt}\left(\boldsymbol{V_a} \cdot d\boldsymbol{l}\right) - \boldsymbol{V_a} \cdot \frac{D_a}{Dt}(d\boldsymbol{l})$$

or after observing that since l is a position vector,

$$\frac{D_a \boldsymbol{l}}{Dt} = \boldsymbol{V_a},$$

$$\frac{D_a \boldsymbol{V_a}}{Dt} \cdot d\boldsymbol{l} = \frac{D}{Dt}\left(\boldsymbol{V_a} \cdot d\boldsymbol{l}\right) - \boldsymbol{V_a} \cdot d\boldsymbol{V_a} \qquad (4.2)$$

Substituting (4.2) into (2.8)'' leads to

$$\frac{D_a}{Dt}(\boldsymbol{V_a} \cdot d\boldsymbol{l}) - \boldsymbol{V_a} \cdot d\boldsymbol{U_a} = -\frac{1}{\rho}\nabla p \cdot d\boldsymbol{l} - \nabla \phi \cdot d\boldsymbol{l}$$

Taking a close line integral of the above equation gives

$$\underset{(1)}{\oint \frac{D_a}{Dt}(\boldsymbol{V_a} \cdot d\boldsymbol{l})} - \underset{(2)}{\oint \boldsymbol{V_a} \cdot d\boldsymbol{V_a}} = \underset{(3)}{-\oint \frac{\nabla p \cdot d\boldsymbol{l}}{\rho}} - \underset{(4)}{\oint \nabla \phi \cdot d\boldsymbol{l}}$$

Term (1): $\oint \frac{D_a}{Dt}(\boldsymbol{V_a} \cdot d\boldsymbol{l}) = \frac{D_a}{Dt}\oint (\boldsymbol{V_a} \cdot d\boldsymbol{l}) = \frac{D_a C_a}{Dt} = \frac{DC_a}{Dt}$

Term (2): $-\oint \boldsymbol{V_a} \cdot d\boldsymbol{V_a} = -\frac{1}{2}\oint d(\boldsymbol{V_a} \cdot \boldsymbol{V_a}) = 0$

(because close line integral of an exact differential is 0.)

Term (3): $-\oint \frac{\nabla p \cdot d\boldsymbol{l}}{\rho} = -\oint \frac{dp}{\rho}$

Term (4): $-\oint \nabla\phi \cdot d\boldsymbol{l} = -\oint d\phi = 0$.

Thus, we obtain the circulation theorem:

$$\frac{DC_a}{Dt} = -\oint \frac{dp}{\rho} \qquad (4.3)$$

The term on the right hand side is called "solenoidal term". The physical meaning of the solenoidal term will be explained later. (1/19/17)

2.2 Kelvin's Circulation Theorem

For a barotropic fluid, $\rho = \rho(p,T) = \rho(p)$, there is no temperature difference on an isobaric (pressure) surface. This lead to

$$\frac{DC_a}{Dt} = -\oint \frac{dp}{\rho} = 0.$$

e.g., suppose $\rho = \rho(p) = ap$, then

$$\oint \frac{dp}{\rho} = \oint \frac{dp}{ap} = \frac{1}{a}\oint \frac{dp}{p} = \frac{1}{a}\oint d(\ln p) = \frac{1}{a}\ln p\Big|_{p_o}^{p_o} = 0.$$

(Note that the closed line integral of an exact differential is 0.)

In other words, in a barotropic atmosphere or fluid, the absolute circulation is conserved following the motion, i.e.

$$\frac{DC_a}{Dt} = 0.$$

This is called the Kelvin's circulation theorem.

It can be shown (in homework problem) that Kelvin's circulation theorem is analogous to the conservation of angular momentum.

Recall that

Linear momentum: $P_{linear} = mv$

Angular momentum: $L = I\Omega$ where I is the moment of inertia and Ω is the angular velocity.

2.3 Bjerknes Circulation Theorem

For meteorological applications, it is more convenient to use the relative circulation.

Bjerknes extends Kelvin circulation theorem to the "Bjerknes circulation theorem".

Recall

$$\boldsymbol{V}_a = \boldsymbol{V} + \boldsymbol{\Omega} \times \boldsymbol{r} \qquad \text{(2.5) (Holton)}$$

Taking $\cdot\, \boldsymbol{dl}$ and integrate along a closed contour on earth's surface gives

$$\oint \boldsymbol{V}_a \cdot d\boldsymbol{l} = \oint \boldsymbol{V} \cdot d\boldsymbol{l} + \oint (\boldsymbol{\Omega} \times \boldsymbol{r}) \cdot d\boldsymbol{l}$$

Absolute Circulation   Relative Circulation

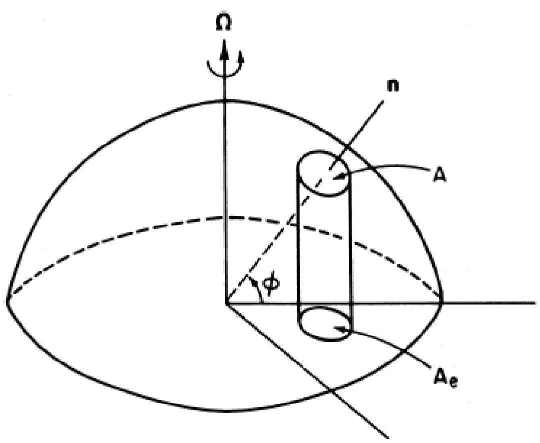
After some manipulation of the second term on the right-hand side, the above equation can be rewritten as

$$C_a = C + 2\Omega A \sin\bar{\phi} = C + 2\Omega A_e,$$

Here A_e defined as

$$A_e = A \sin\bar{\phi} \, .$$

is the projection of A on equatorial plane as shown below:



Taking integration of the above equation involving C_a yields

$$\frac{DC_a}{Dt} = \frac{DC}{Dt} + 2\Omega \frac{D(A\sin\bar{\phi})}{Dt}, \text{ or}$$

$$\frac{DC}{Dt} = \frac{DC_a}{Dt} - 2\Omega \frac{D(A\sin\bar{\phi})}{Dt}.$$

Inserting the circulation theorem into the above equation gives the Bjerknes circulation theorem:

$$\frac{DC}{Dt} = -\oint \frac{dp}{\rho} - 2\Omega \frac{D}{Dt}(A\sin\bar{\phi}) \qquad (4.5)$$

For a barotropic atmosphere (no temperature variation on an isobaric surface), Eq. (4.5) reduces to

$$\frac{DC}{Dt} = -2\Omega \frac{D}{Dt}(A\sin\bar{\phi}).$$

Integrating the above equation from time 1 to 2 leads to

$$\int_1^2 \frac{DC}{Dt}dt = -2\Omega \int_1^2 \frac{D}{Dt}(A\sin\bar{\phi})dt,$$

or

$$C_2 - C_1 = -2\Omega(A_2 \sin\bar{\phi}_2 - A_1 \sin\bar{\phi}_1). \qquad (4.6)$$

That is, circulation changes if the area of the fluid chain or the latitude changes.

- Applications of Bjerknes circulation theorem.

  Example 1: Consider an area, say A, originally located at equator and moved to North Pole without changing its area. Estimate the final circulation C.

Example 2: Sea-breeze circulation

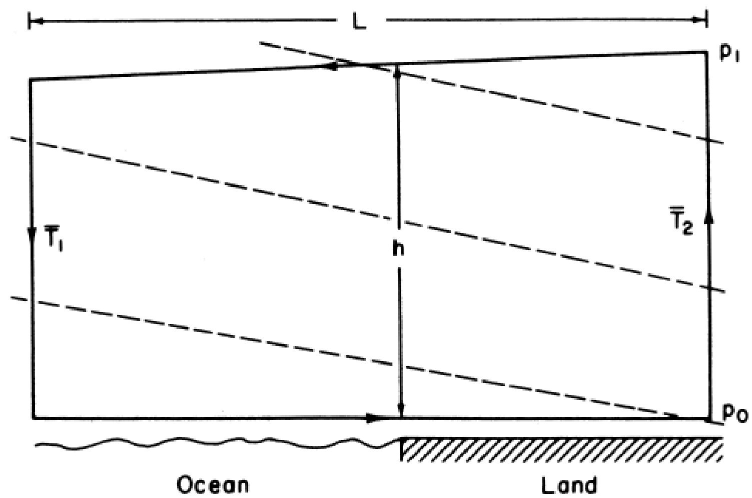


Fig. 4.3 Application of the circulation theorem to the sea breeze problem. The closed heavy solid line is the loop about which the circulation is to be evaluated. Dashed lines indicate surfaces of constant density.

Note that the isothermal surfaces are tilted in opposite way of the density surface.

$$\frac{DC_a}{Dt} = -\oint \frac{dp}{\rho} = -\oint RT\frac{dp}{p} = -\oint RT\, d(\ln p)$$

$$= -\int_a^b RT\, d(\ln p) - \int_b^c RT\, d(\ln p) - \int_c^d RT\, d(\ln p) - \int_d^a RT\, d(\ln p)$$

Where *a* denotes the lower left corner, *b* the lower right corner, *c* upper right corner, and d the upper left corner.

Assuming the isobaric (pressure) surface is nearly horizontal, then the 1st and 3rd terms are approximately 0.

$$\frac{DC_a}{Dt} = -\int_b^c RT\, d(\ln p) - \int_d^a RT\, d(\ln p) = -R\overline{T}_2 \int_b^c d(\ln p) - R\overline{T}_1 \int_d^a d(\ln p)$$

$$\frac{DC_a}{Dt} = -R\overline{T}_2 \ln\frac{p_c}{p_b} - R\overline{T}_1 \ln\frac{p_a}{p_d} = R(\overline{T}_2 - \overline{T}_1)\ln\frac{p_o}{p_1}\ .$$

- Physical meaning of the solenoidal term in Holton's Eq. (4.3) and its application to a sea-breeze circulation.