

## Lecture 17 The Clausius-Clapeyron Equation

(Ref.: Sec.4.5 of Hess)

- In this lecture, we will derive an important equation, the Clausius-Clapeyron equation, which calculates the change of the saturation vapor pressure with temperature ( $de_s/dT$ ) during a phase change.
- Based on observations and experiments, a phase change is reversible and isothermal.

The latent heat for a reversible process may be calculated by

$$L_{12} = \int_1^2 dq = \int_1^2 T ds = T(s_2 - s_1). \quad (17.1)$$

Thus, in order to relate  $L_{12}$  to saturation pressure, we need to relate entropy  $s_2$  and  $s_1$  to saturation pressure.

➤ Consider the following cyclic process in an  $\alpha$ - $e$  diagram.

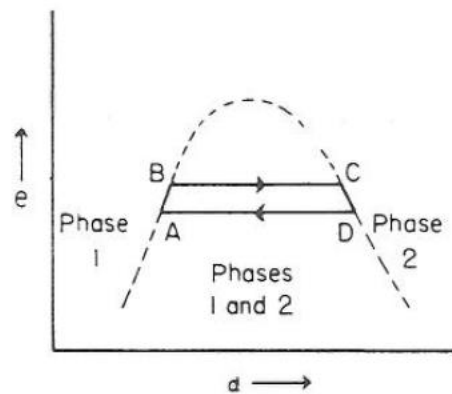


Fig. 4.2

**A→B:** A slight increase in temperature,  $\Delta T$ , without any change in phase leading to state B:  $(e_s + \Delta e_s, \alpha_1 + \Delta \alpha_1, T + \Delta T)$

**B→C:** an isothermal phase change leading to state C:  $(e_s + \Delta e_s, \alpha_2 + \Delta \alpha_2, T + \Delta T)$

**C→D:** A slight decrease in temperature,  $-\Delta T$ , without any change of phase, leading to state D:  $(e_s, \alpha_2, T)$

**D→A:** an isothermal change of phase back to state A:  $(e_s, \alpha_1, T)$

➤ Apply the first law to the above processes,

$$dq = Tds = du + e_s d\alpha.$$

Thus, for the whole cyclic process,

$$\oint Tds = \oint e_s d\alpha, \quad (17.2)$$

because for a closed line integral  $du$  is zero ( $du$  is an exact differential).

The right side of (17.2) is nothing but the area enclosed by the curves, which is approximately equal to  $(\alpha_2 - \alpha_1)\Delta e_s$ .

Thus, we have

$$\oint Tds = (\alpha_2 - \alpha_1)\Delta e_s . \quad (17.3)$$

Since  $d(Ts)$  is an exact differential, we have

$$\oint Tds = -\oint s dT .$$

We now evaluate this integral,  $-\oint s dT$ , along each of the four parts of the cycle.

**A→B**: no heat added, thus  $s$  is a constant.

$$-\int_A^B s dT = -s_1\Delta T .$$

**B→C**: Isothermal process,

$$-\int_B^C s dT = 0 .$$

**C→D**: no heat added, similar to **A→B**,

$$-\int_C^D s dT = s_2\Delta T .$$

**D→A**: Isothermal process,

$$-\int_D^A s dT = 0 .$$

As a result,

$$\oint T ds = (s_2 - s_1)\Delta T. \quad (17.4)$$

Combining Eqs. (17.3) and (17.4) leads to

$$(s_2 - s_1)\Delta T = (\alpha_2 - \alpha_1)\Delta e_s$$

or

$$s_2 - s_1 = (\alpha_2 - \alpha_1) \frac{de_s}{dT}.$$

Substituting the above equation into (17.1) leads to

$$L_{12} = (\alpha_2 - \alpha_1) T \frac{de_s}{dT},$$

or

$$\frac{de_s}{dT} = \frac{L_{12}}{T(\alpha_2 - \alpha_1)}. \quad (17.5)$$

This is called **Clausius-Clapeyron equation**. If  $L_{12}$ ,  $\alpha_1$ , and  $\alpha_2$  are known functions of  $T$ , then (17.5) can be integrated to obtain a relation between saturation vapor pressure and  $T$ .

It gives the slope of the curves of saturation vapor pressure versus temperature as a function of the latent heat, temperature and the difference in specific volume of the two phases.

- We will **consider two special cases: evaporation and sublimation**. Since for these two cases,  $\alpha_2$  is the specific

volume for water vapor which is much greater than  $\alpha_l$  (for liquid water or ice), and  $L_{12}$  is nearly a constant for each processes.

Thus, the Clausius-Clapeyron equation may be approximated by

$$\frac{de_s}{dT} = \frac{L_{12}}{T\alpha_2}. \quad (17.6)$$

Using the equation of state for ideal gases,

$$e_s\alpha_2 = R_v T \text{ or } \alpha_2 = \frac{R_v T}{e_s},$$

we get

$$\frac{de_s}{e_s} = \frac{L_{12}}{R_v} \frac{dT}{T^2}. \quad (17.7)$$

Therefore,

$$\ln\left(\frac{e_s}{e_{s0}}\right) = \frac{L_{12}}{R_v} \left(\frac{1}{T_0} - \frac{1}{T}\right). \quad (17.8)$$

where  $e_{s0}$  is the saturation vapor pressure at  $T_0$ .

For evaporation,  $e_{s0}=6.11$  mb and  $T_0=273$  K, which implies

$$\ln\left(\frac{e_s}{6.11 \text{ mb}}\right) = \frac{L_{\text{evap}}}{R_v} \left(\frac{1}{273} - \frac{1}{T}\right). \quad (17.9)$$

or

$$e_s = 6.11 \text{ mb} \exp\left[\frac{L_{\text{evap}}}{R_v} \left(\frac{1}{273} - \frac{1}{T}\right)\right]. \quad (17.10)$$

For sublimation,

$$\ln\left(\frac{e_s}{6.11 \text{ mb}}\right) = \frac{L_{sub}}{R_v} \left(\frac{1}{273} - \frac{1}{T}\right), \quad (17.11)$$

or

$$e_s = 6.11 \text{ mb} \exp\left[\frac{L_{sub}}{R_v} \left(\frac{1}{273} - \frac{1}{T}\right)\right]. \quad (17.12)$$

**Example:** Calculate the change in the melting point of ice if the pressure is increased from 1 to 2 atm, given that

$$\alpha_i = 1.0908 \times 10^{-3} \text{ m}^3 \text{ kg}^{-1}, \quad \alpha_w = 1.001 \times 10^{-3} \text{ m}^3 \text{ kg}^{-1}$$

$$L_{melt} = 3.34 \times 10^5 \text{ J kg}^{-1} \text{ at } 0^\circ \text{C}.$$

**Solution:** Using Clausius-Clapeyron equation,

$$\begin{aligned} dT &= T(\alpha_w - \alpha_i) \frac{dp}{L_{melt}} \\ &= 273 \times (1.001 - 1.0908) \times 1.013 \times 10^5 / 3.34 \times 10^5 \\ &= -0.00744 \text{ deg} \end{aligned}$$

Therefore, an increase in pressure of 1 atm decreases the melting point of ice by about 0.007 deg. Usually, the melting point increases with increasing pressure. But ice is unusual because  $\alpha_w < \alpha_i$ .

- Equations (17.9) and (17.11) plus a similar equation for melting allow us to plot the curves of saturation pressure versus temperature. These curves describe the  $T$ - $e$  relations during phase changes ( $e=e_s$ ).

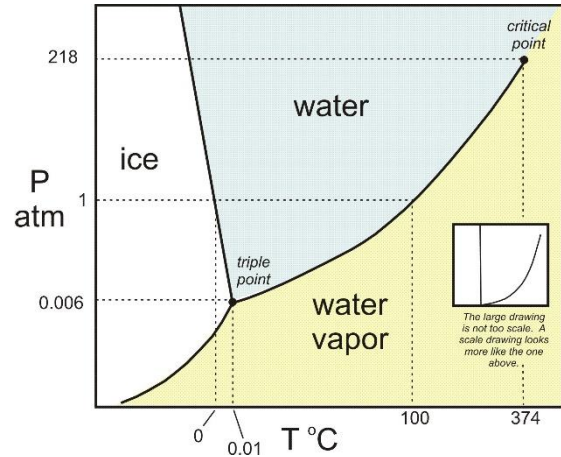
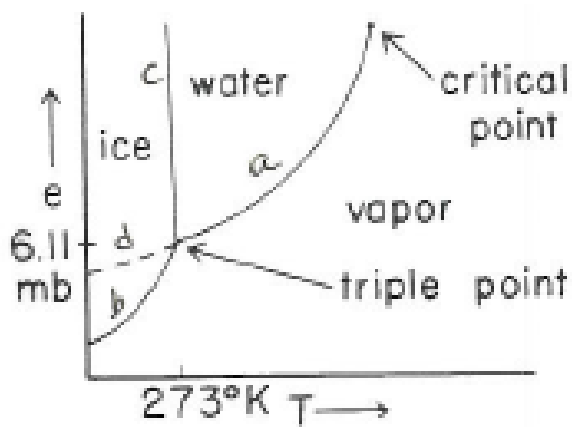


Figure 4.3 (Hess):  $T, e$  phase diagram for water substance.

[Ref]

All curves begin at the triple point.

(a) **Evaporation curve a**: curves upward exponentially to the right according to Eq. (17.9) ( $T > 273$  K, and  $e_s > 6.11$  mb). Along this curve, water and vapor are in equilibrium until it reaches the critical point ( $T = 374$  K) where only vapor can exist. **At some point along this curve, water starts to boil.** That is the point when  $e_s = p_{atm}$ .

(b) **Sublimation curve b**: curves downward exponentially to the left of the triple point ( $T < 273$  K,  $e_s < 6.11$  mb). This curve is steeper than the evaporation curve because  $L_{sub} > L_{evap}$ .

(c) **Melting curve c**: since  $\alpha_w - \alpha_i$  is almost zero (a very small negative number),  $de_s/dT$  is almost  $-\infty$ , based on the Clausius-Clapyeron equation. **Thus the curve is almost vertical**, but tilts very slightly to the left.

(d) **Supercooled water d**: When water which does not freeze below  $273$  K ( $0^\circ\text{C}$ ), it is called **supercooled water**.

For supercooled water, within a certain temperature range of  $T < 0^{\circ}\text{C}$ ,  $e_{sw} > e_{si}$  (curve d).

Therefore, in a cloud with mixture of ice crystals and supercooled water (cloud) droplets, **water vapor will first sublime on the ice** because  $e_{si}$  is smaller than  $e_{sw}$ .

In other words, in the **competition for water vapor**, ice will win over supercooled water. This leads to the well-known **Bergeron-Findeison process**.

That is, **ice crystal in a cloud of supercooled water droplets will grow faster than the water droplets**.



- Since for each temperature ( $T$ ), there is a relation (2-D curve) between  $e$  and  $\alpha$ , for all possible  $T$ , these curves become a three-dimensional surface (Figure 4.4)

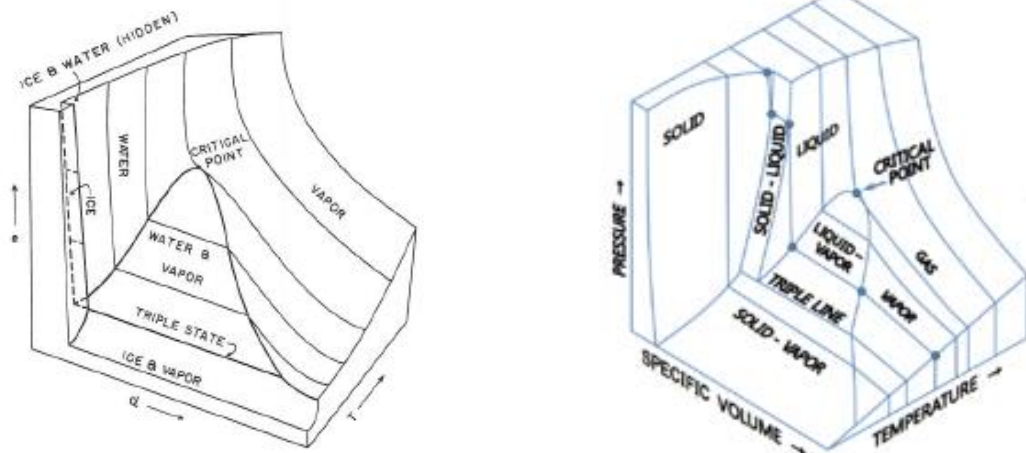


Figure 4.4: (Hess) 3D thermodynamic surface of water substance.

[[Reference](#)]