

Ch.2 Scale Analysis of Equations of Motion

2.1 Scale Analysis

Objectives of the scale analysis:

- (1) To simplify the mathematics by eliminating insignificant terms in the equations, and
- (2) To filter out the unwanted disturbances, such as sound waves and gravity waves in numerical weather prediction (NWP) simulations.

Definitions of atmospheric scales (Lin 2007 – Mesoscale Dynamics, Cambridge U. Press):

“Based on radar observations of storms, atmospheric motions can be categorized into the following three scales (Ligda 1951):

- (a) **Synoptic (large) scale:** $1000 \text{ km} < L$
- (b) **Mesoscale:** $20 \text{ km} < L < 1000 \text{ km}$
- (c) **Microscale:** $L < 20 \text{ km}$

The atmospheric motions have also been categorized into 8 separate scales (Orlanski 1975; Table 1.1):

- (a) **Macroscale:** $2000 \text{ km} < L < 10,000 \text{ km}$
macro- α ($10,000 \text{ km} < L$) [planetary scale]
macro- β ($2000 \text{ km} < L < 10,000 \text{ km}$)
[synoptic scale to planetary scale]
- (b) **Mesoscale:** $2 \text{ km} < L < 2000 \text{ km}$
meso- α ($200 \text{ km} < L < 2000 \text{ km}$)
meso- β ($20 \text{ km} < L < 200 \text{ km}$)
meso- γ ($2 \text{ km} < L < 20 \text{ km}$)

- (c) **Microscale: $2 \text{ m} < L < 2 \text{ km}$**
micro- α ($200 \text{ m} < L < 2 \text{ km}$)
micro- β ($20 \text{ m} < L < 200 \text{ m}$)
micro- γ ($2 \text{ m} < L < 20 \text{ m}$) scales

Based on theoretical considerations, the following different scales for atmospheric motions can be defined (Emanuel and Raymond 1984):

- (a) **synoptic (large or macro) scale**: for motions which are quasi-geostrophic and hydrostatic,
(b) **mesoscale**: for motions which are non-quasi-geostrophic and hydrostatic, and
(c) **microscale**: for motions which are non-geostrophic, nonhydrostatic, and turbulent

Table 1.1 Atmospheric scale definitions. (Adapted after Thunis and Borstein 1996)
 (From Lin 2007 – Mesoscale Dynamics, Cambridge U. Press)

Horizontal Scale	Lifetime	Stull (1988)	Pielke (2002)	Orlanski (1975)	Thunis and Bornstein (1996)	Atmospheric Phenomena
10 000 km	1 month	Macro	Synoptic Regional	Macro- α	Macro- α	General circulation, long waves
				Macro- β	Macro- β	Synoptic cyclones
2000 km	1 week	Meso	Meso	Meso- α	Macro- γ	Fronts, hurricanes, tropical storms, short cyclone waves, mesoscale convective complexes
200 km	1 day			Meso- β	Meso- β	Mesocyclones, mesohighs, supercells, squall lines, inertia-gravity waves, cloud clusters, low-level jets, thunderstorm groups, mountain waves, sea breezes
20 km	1 h	Micro	Micro	Meso- γ	Meso- γ	Thunderstorms, cumulonimbi, clear-air turbulence, heat island, macrobursts
2 km				Micro- α	Meso- δ	Cumulus, tornadoes, microbursts, hydraulic jumps
200 m	30 min	Micro- δ	Micro	Micro- β	Micro- β	Plumes, wakes, waterspouts, dust devils
20 m	1 min			Micro- γ	Micro- γ	Turbulence, sound waves
2 m	1 s			Micro- δ		

(a) Horizontal momentum equation

From Ch. 1, we have derived the equation of motion

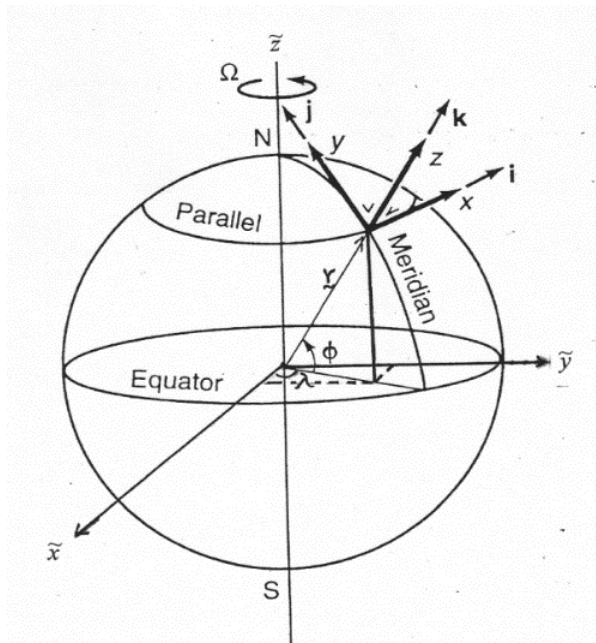
$$\frac{DV}{Dt} = -\frac{1}{\rho} \nabla p - 2\boldsymbol{\Omega} \times \mathbf{V} + \mathbf{g} + \mathbf{F}_r, \quad (2.8)$$

Coriolis force

where $\mathbf{g} = \mathbf{g}^* + \Omega^2 \mathbf{R}$ is the **effective gravity**.

Equation (2.8) can be transformed into scalar components in spherical coordinates (Sec. 2.3, Holton)

- **Coordinate systems on the Earth's Rotating Frame of Reference**



Equations of motion on the local coordinates with earth curvature included:

$$\frac{Du}{Dt} = 2\Omega v \sin \phi - 2\Omega w \cos \phi - \frac{uw}{a} - \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{uv \tan \phi}{a} + \nu \nabla^2 u \quad (2.19)$$

$$\frac{Dv}{Dt} = -2\Omega u \sin \phi - \frac{vw}{a} - \frac{u^2 \tan \phi}{a} - \frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \nabla^2 v \quad (2.20)$$

$$\frac{Dw}{Dt} = 2\Omega u \cos \phi + \frac{u^2 + v^2}{a} - \frac{1}{\rho} \frac{\partial p}{\partial z} - g + \nu \nabla^2 w \quad (2.21)$$

In this course, we will focus on **midlatitude synoptic systems**, which have the following characteristic scales:

$U \sim 10 \text{ ms}^{-1}$	horizontal velocity scale ($u, v \sim U$)
$W \sim 1 \text{ cm s}^{-1}$ or 10^{-2} ms^{-1}	vertical velocity scale ($w \sim W$)
$L \sim 1000 \text{ km}$ or 10^6 m	horizontal length scale ($\Delta x, \Delta y \sim L$)
$L_z \sim 10 \text{ km}$ or 10^4 m	vertical length scale due to motion ($\Delta z \sim L_z$)
	($L_z \sim 1 \text{ km}$ for shallow vertical-motion systems)
$H \sim 10 \text{ km}$ or 10^4 m	vertical length scale due to atmospheric structure
$(\Delta p)_{x,y} \sim 10 \text{ mb}$ or 10^3 Pa	hori. pressure perturbation scale
$T \sim L/U = 10^5 \text{ s}$	time scale
$\rho_o \sim 1 \text{ kg m}^{-3}$	density scale
$f_o \sim 10^{-4} \text{ s}^{-1}$	Coriolis parameter ($\sim 2\Omega \sin 45^\circ$)
$a \sim 10^7 \text{ m}$ ($\sim 6400 \text{ km}$)	Earth radius
$\nu \sim 10^{-5} \text{ m}^2 \text{ s}^{-1}$	coefficient of molecular friction

Scale analysis of the horizontal momentum equations:

$$\frac{Du}{Dt} - 2\Omega v \sin \phi + 2\Omega w \cos \phi + \frac{uw}{a} - \frac{uv \tan \phi}{a} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \nabla^2 u \quad (2.19)$$

$$\text{Scales} \quad \frac{U^2}{L} \left(\frac{UW}{L_z} \right) \quad f_o U \quad f_o W \quad \frac{UW}{a} \quad \frac{U^2}{a} \quad \frac{\delta p}{\rho_o L} \quad \frac{\nu U}{L^2} \left(\frac{\nu U}{L_z^2} \right)$$

Magnitude (in m/s²)

10⁻⁴ (10⁻⁵) 10⁻³ 10⁻⁶ 10⁻⁸ 10⁻⁵ 10⁻³ 10⁻¹⁶ (10⁻¹²)

(1) Geostrophic approximation [keeping terms of O(10⁻³)]

Keeping the terms with the highest order of magnitude (10⁻³) gives the **geostrophic wind**

$$-fv_g = -\frac{1}{\rho} \frac{\partial p}{\partial x}, \quad (2.22a)$$

where $f = 2\Omega \sin \phi$ is called the “**Coriolis parameter**” and v_g is called the **geostrophic wind**.

Similarly, the geostrophic wind in y direction can be derived

$$fu_g = -\frac{1}{\rho} \frac{\partial p}{\partial y}. \quad (2.22b)$$

Equations (2.22a) and (2.22b) can be written in vector form

$$f\mathbf{V}_g = \mathbf{k} \times \frac{1}{\rho} \nabla p \quad (2.23)$$

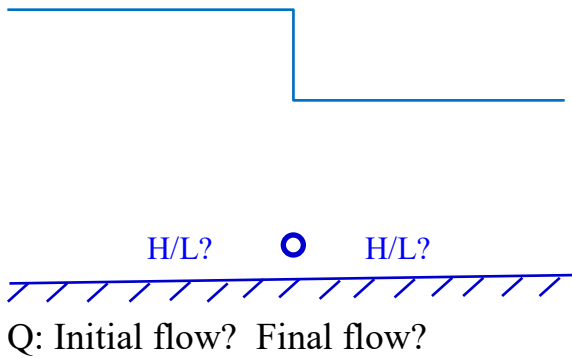
where $\mathbf{V}_g = u_g \mathbf{i} + v_g \mathbf{j}$ is the geostrophic wind velocity.

Characteristics of the geostrophic wind:

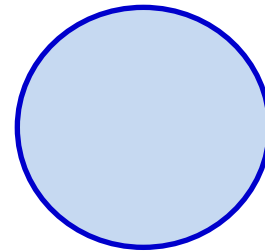
- (a) \mathbf{V}_g approximates the actual wind to within 10 – 15%
- (b) \mathbf{V}_g // isobars leaving the low to the left in Northern Hemisphere.
- (c) \mathbf{V}_g is larger for smaller-spacing isobars.
- (d) \mathbf{V}_g is time-independent.

➤ Examples of geostrophic adjustment problem

(a) How does the fluid response to a large-scale near-surface disturbance?

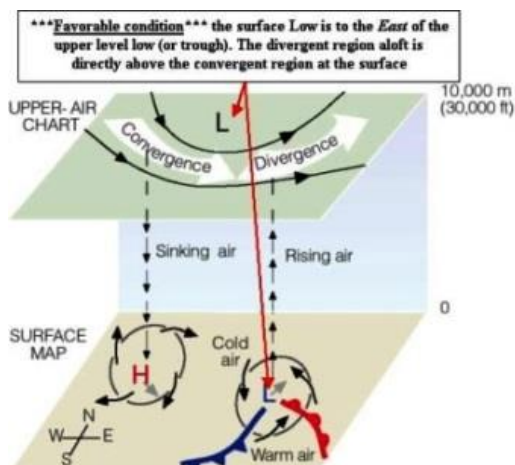


(b) How does the air adjust to a large-scale cold dome?



Q: Initial flow? Final flow?

(c) Surface flow adjust to upper air disturbance



(d) Tropical system development

Stages of Development

Depression	Tropical Storm	Hurricane

- **Tropical Disturbance:** Winds weak and unorganized
- **Tropical Depression:** Winds less than 39 mph
- **Tropical Storm:** Winds 39 to 74 mph
- **Cyclone:** Winds greater than 74 mph

(2) Approximate prognostic equations [keeping terms equal or higher than $O(10^{-4})$]

If we keep all terms of $O(10^{-4})$ and higher, then we have

$$\frac{Du}{Dt} = fv - \frac{1}{\rho} \frac{\partial p}{\partial x} \quad (2.24)$$

$$\frac{Dv}{Dt} = -fu - \frac{1}{\rho} \frac{\partial p}{\partial y} \quad (2.25)$$

Inertial Force (acceleration)
Coriolis Force
PGF Force

Equations (2.24) and (2.25) reduce to (2.22a) and (2.22b), respectively, whenever the first terms on the left hand side are very small compared to other terms, e.g.

$$\frac{\text{Inertial Force}}{\text{Coriolis Force}} = \frac{Du/Dt}{fv} \approx \frac{U/T}{fU} = \frac{1}{fT} \ll 1$$

where T is the time scale.

For an Eulerian frame of reference, the time scale can be calculated by $T = L/U$. Substituting it into the above equation leads to

$$R_o \equiv \frac{U}{fL} \ll 1$$

R_o is called the Rossby number.

Considering an air parcel following the motion, a Lagrangian Rossby number may be defined as

$$R_o = \frac{1}{fT} = \frac{1}{f(2\pi R/V_T)} = \frac{V_T}{2\pi fR},$$

where R is the radius of a circular motion or **radius of local curvature**. Sometimes the **Lagrangian Rossby number** is defined as

$$R_o = \frac{\omega}{f} = \frac{2\pi}{fT} = \frac{V_T}{fR}.$$

where that ω is defined as $2\pi/T$ and referred to as the **angular frequency**, i.e. the frequency is measured by angle, instead of by cycle, which is different from what you've learned from General Physics (Lin 2007).

Note that when you compare the inertial force term to the viscous force term of the equation of motion, Eq. (2.19), it leads to the definition of the **Reynolds number** $Re = LU / \nu$.

Phenomenon	Time scale	Lagrangian R_o ($\approx \omega / f = 2\pi / fT$)
Tropical cyclone	$2\pi R / V_T$	V_T / fR
Inertia-gravity waves	$2\pi / N$ to $2\pi / f$	N / f to 1
Sea/land breezes	$2\pi / f$	1
Cumulus clouds	$2\pi / N_w$	N_w / f
Kelvin-Helmholtz waves	$2\pi / N$	N / f
PBL turbulence	$2\pi h / U^*$	U^* / fh
Tornadoes	$2\pi R / V_T$	V_T / fR


where

- R = radius of maximum wind scale
- ω = frequency
- T = time scale
- V_T = maximum tangential wind scale
- f = Coriolis parameter
- N = buoyancy (Brunt-Vaisala) frequency
- N_w = moist buoyancy (Brunt-Vaisala) frequency
- U^* = scale for friction velocity
- h = scale for the depth of planetary boundary layer.

(b) Vertical momentum equation (see additional note)

$$\frac{Dw}{Dt} - 2\Omega u \cos\phi - \frac{u^2 + v^2}{a} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g + \nu \nabla^2 w$$

Scales	$\frac{UW}{L} \left(\text{or } \frac{W^2}{L_z} \right)$	$f_o U$	$\frac{U^2}{a}$	$\frac{\delta p_z}{\rho_o H}$	g	$\frac{\nu W}{L_z^2} \left(\text{or } \frac{\nu W}{L^2} \right)$
Magnitude (in m/s ²)	$10^{-7} (10^{-8})$	10^{-3}	10^{-5}	10	10	$10^{-15} (10^{-19})$



From basic atmospheric structure

Keeping the terms of largest order of magnitude, i.e. O(10), leads to:

(1) Hydrostatic equation

$$-\frac{1}{\rho} \frac{\partial p}{\partial z} - g = 0$$

Under this approximation, the gravitational force is balanced by the vertical PGF approximately. The approximation is called “**hydrostatic approximation**”.

Note that it is misleading to merely show the vertical acceleration (Dw/Dt) term is much smaller than the vertical PGF term. **It is necessary to compare it to the perturbation PGF.**

$$\frac{\partial p'}{\partial z} = -\rho' g. \tag{2.28}$$

[Reading assignment] See Holton and Hakim’s [section 2.4.3](#) for details.

(2) Important terminologies and concepts

- Geopotential
- Geopotential height
- Hypsometric equation
- Scale height

(a) Geopotential

As discussed earlier, **geopotential** is defined as the **work done** when an air parcel of unit mass (1 kg) is lifted from sea level to a certain height z

(AMS Glossary of Meteorology:

<http://amsglossary.allenpress.com/glossary/search?id=geopotential-height1>)

$$\phi = \int_0^z g dz$$

(b) Geopotential height

Geopotential height Z is defined as the height of a given point in the **atmosphere** in units proportional to the **potential energy** of unit mass (**geopotential**) at this height relative to **sea level**.

$$Z = \frac{1}{g_0} \int_0^z g dz .$$

- The actual height of an air parcel and the geopotential height are numerically interchangeable for most meteorological purposes.
- Higher (Lower) $Z \Leftrightarrow$ higher (lower) pressure (**give example here**)

(c) Hypsometric equation

Integrating the equation of geopotential definition from z_1 to z_2 and substituting the hydrostatic equation into it leads to (reading assignment)

$$\phi(z_2) - \phi(z_1) = g_o(Z_2 - Z_1) = R \int_{p_1}^{p_2} T \, d(\ln p). \quad (1.21)$$

Equation (1.21) can be approximated by

$$\phi(z_2) - \phi(z_1) = g_o(Z_2 - Z_1) = -R\bar{T} \ln \frac{p_1}{p_2}$$

Thus, the physical meaning of the hypsometric equation is that the depth of an atmospheric layer is proportional to the mean layer temperature.

(d) Scale height

The height where the sea-level density (ρ_o) is reduced to its e-folding value ($\rho_o e^{-1}$). Note that approximately the air density is reduced exponentially

$$\rho(z) = \rho_o e^{-z/H}.$$

Thus, $z = H$ is the scale height.