

Chapter 4

Elementary Applications of the Basic Equations

4.1 Basic Equations in Isobaric Coordinates

(Ref: Holton Sec. 3.1)

➤ **The Horizontal Momentum Equation**

The approximate horizontal momentum equations (2.24) and (2.25) may be written in vectoral form as

$$\frac{Du}{Dt} = fv - \frac{1}{\rho} \frac{\partial p}{\partial x} \quad (2.24)$$

$$\frac{Dv}{Dt} = -fu - \frac{1}{\rho} \frac{\partial p}{\partial y} \quad (2.25)$$

Inertial Force Coriolis PGF

[1st term: Also called total rate of change following the motion; total derivative, material derivative]

$$\frac{D\mathbf{V}}{Dt} + f \mathbf{k} \times \mathbf{V} = -\frac{1}{\rho} \nabla p \quad (3.1)$$

where $\mathbf{V} = u\mathbf{i} + v\mathbf{j}$ is the horizontal velocity vector.

Substituting the following gradient force in isobaric coordinates,

$$-\frac{1}{\rho} \left(\frac{\partial p}{\partial x} \right)_z = - \left(\frac{\partial \phi}{\partial x} \right)_p, \quad (1.20)$$

$$-\frac{1}{\rho} \left(\frac{\partial p}{\partial y} \right)_z = - \left(\frac{\partial \phi}{\partial y} \right)_p \quad (1.21)$$

into (3.1) leads to

$$\frac{DV}{Dt} + f \mathbf{k} \times \mathbf{V} = -\nabla_p \phi \quad (3.2)$$

where ∇_p is the horizontal gradient operator applied with pressure held constant.

Because p is the independent vertical coordinate, we must expand the total derivative as

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{Dx}{Dt} \frac{\partial}{\partial x} + \frac{Dy}{Dt} \frac{\partial}{\partial y} + \frac{Dp}{Dt} \frac{\partial}{\partial p} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + \omega \frac{\partial}{\partial p} \quad (3.3)$$

Here $\omega = Dp / Dt$ is called the *omega vertical motion* which is defined as the pressure change following the motion, equivalent to $w = Dz / Dt$ in height coordinates.

Note that for synoptic motions, $\omega \approx -\rho g w$.

- From (3.2), the *geostrophic relation in isobaric coordinates* can be written as

$$f \mathbf{V}_g = \mathbf{k} \times \nabla_p \phi \quad (3.4)$$

or in scalar form

$$f u_g = -\frac{\partial \phi}{\partial y}, \quad (3.4a)$$

$$f v_g = \frac{\partial \phi}{\partial x}. \quad (3.4b)$$

Note there is no density present in (3.4).

In addition, on an f -plane (i.e., f is constant), we have

$$\nabla_p \cdot \mathbf{V}_g = 0$$

That is, there is **no divergence for the geostrophic flow (non-divergent)**.

- The **continuity equation in the isobaric coordinates** can be derived directly from Eq. (2.31)

$$\frac{1}{\rho} \frac{D\rho}{Dt} + \nabla \cdot \mathbf{V} = 0, \quad (2.31)$$

But it is easier to derive the isobaric form by considering a Lagrangian control volume $\delta V = \delta x \delta y \delta z$ and $\delta p = -\rho g \delta z$. The mass, $\delta M = \rho \delta V = -\delta x \delta y \delta p / g$, is conserved following the motion,

$$\frac{1}{\delta M} \frac{D}{Dt} \delta M = \frac{g}{\delta x \delta y \delta p} \frac{D}{Dt} \left(\frac{\delta x \delta y \delta p}{g} \right) = 0.$$

Applying the chain rule, we obtain

$$\frac{1}{\delta x} \delta \left(\frac{Dx}{Dt} \right) + \frac{1}{\delta y} \delta \left(\frac{Dy}{Dt} \right) + \frac{1}{\delta p} \delta \left(\frac{Dp}{Dt} \right) = 0$$

which gives us

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)_p + \frac{\partial \omega}{\partial p} = 0. \quad (3.5)$$

- The **thermodynamic energy equation**

Taking the total derivative of the **equation of state**

$$p\alpha = RT \quad (a)$$

Gives

$$p \frac{D\alpha}{Dt} + \alpha \frac{Dp}{Dt} = R \frac{DT}{Dt} \quad (b)$$

Now consider the first law of thermodynamics

$$du + dw = dq$$

or

$$c_v dT + pd\alpha = dq \quad (c)$$

Since $c_p = c_v + R$, (c) can be rewritten as

$$c_p dT - \alpha dp = dq \quad (d)$$

Taking total derivative of (d) gives

$$c_p \frac{DT}{Dt} - \alpha \frac{Dp}{Dt} = J \quad (2.42)$$

where $J = Dq/Dt$ is the diabatic heating rate ($J \text{ kg}^{-1} \text{ s}^{-1}$).

Equation (2.42) may be rewritten as

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} - S_p \omega = \frac{J}{c_p} \quad (3.6)$$

where $J = \frac{Dq}{Dt}$ is the diabatic heating rate and

$$S_p \equiv \frac{RT}{c_p p} - \frac{\partial T}{\partial p} = -\frac{T}{\theta} \frac{\partial \theta}{\partial p} \quad (3.7)$$

or

$$S_p \equiv \frac{\Gamma_d - \Gamma}{\rho g}$$

is a “static stability parameter”.