AST 851 Dynamic Meteorology
Applied Sci&Tech PhD Program
NC A&T State University

Dr. Yuh-Lang Lin
Department of Physics
ylin@ncat.edu
http://mesolab.org

Chapter 4 Elementary Applications of the Basic Equations

4.2 Balanced Flow (ref.: See Sec. 3.2, Holton 4th ed.)

Based on the equation of motion [Eq. (3.2)] in natural coordinates,

$$\frac{D\boldsymbol{V}}{Dt} + f\,\boldsymbol{k}\,x\boldsymbol{V} = -\nabla_p \phi \tag{3.2}$$

We may have the following flow balances:

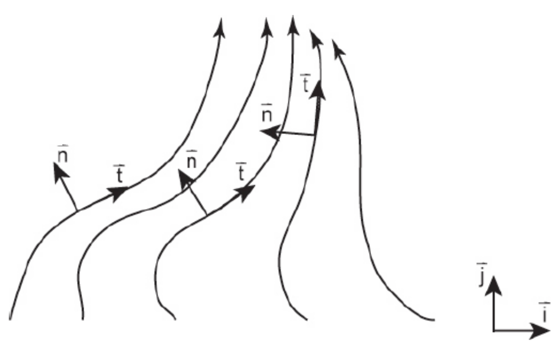
1. Geostrophic flow
2. Inertial flow
3. Cyclostrophic flow
4. Gradient flow

- ### Natural Coordinates

The pressure (or geopotential) and velocity fields in meteorological disturbances are actually related by rather simple approximate balances in natural coordinates.

Define the natural coordinates (***t, n, k***) as unit vectors:

t // local flow
n ⊥ ***t*** perpendicular to ***t*** but pointing to the left



$\boldsymbol{k}$ pointing upward

In this natural coordinate system,

$$\boldsymbol{V} = V\boldsymbol{t} \quad (V \text{ is the wind speed which is always positive})$$

and

$$V = Ds / Dt \,.$$

It can be derived

$$\frac{D\boldsymbol{V}}{Dt} = \frac{D(V\boldsymbol{t})}{Dt} = \boldsymbol{t}\frac{DV}{Dt} + V\frac{D\boldsymbol{t}}{Dt}$$

$$\frac{D\boldsymbol{t}}{Dt} = \frac{D\boldsymbol{t}}{Ds}\frac{Ds}{Dt} = \frac{\boldsymbol{n}}{R}V$$

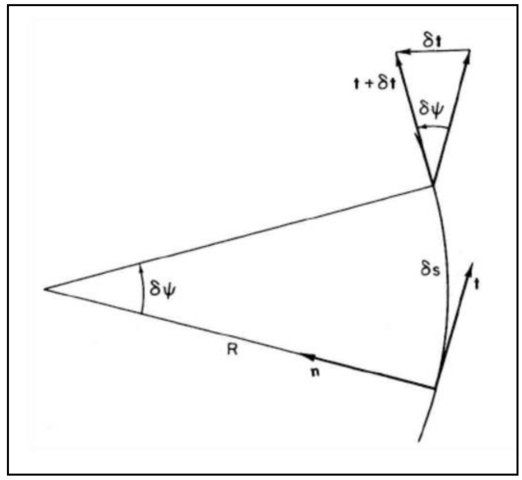


Fig. 3.1: Rate of change of the unit tangent vector ***t*** following the motion.

and

Centripetal acceleration

$$\frac{D\boldsymbol{V}}{Dt} = \boldsymbol{t}\frac{DV}{Dt} + \boldsymbol{n}\frac{V^2}{R} \qquad (3.8)$$

Acceleration following the motion *Tangential acceleration*

Note that centripetal acceleration is exerted by a real force - centripetal force, while centrifugal force is an apparent force.

- In the natural coordinates, we have

$$\boldsymbol{F_{co}} = -f\,\boldsymbol{k} \times \boldsymbol{V} = -fV\boldsymbol{n}$$

$$\boldsymbol{F_p} = -\nabla\phi = -\frac{\partial\phi}{\partial s}\boldsymbol{t} - \frac{\partial\phi}{\partial n}\boldsymbol{n}$$

Substituting $\boldsymbol{F_{co}}$ and $\boldsymbol{F_p}$ into (3.8) leads to

$$\frac{D\boldsymbol{V}}{Dt} = -f\,\boldsymbol{k} \times \boldsymbol{V} - \nabla_p\phi \qquad (3.2)$$

leads to

$$\frac{\partial V}{\partial t}\boldsymbol{t} + \frac{V^2}{R}\boldsymbol{n} = -fV\boldsymbol{n} - \frac{\partial\phi}{\partial s}\boldsymbol{t} - \frac{\partial\phi}{\partial n}\boldsymbol{n},$$

or decompose into $\boldsymbol{t}$ and $\boldsymbol{n}$ components, respectively

$$\boxed{\frac{\partial V}{\partial t} = -\frac{\partial\phi}{\partial s}} \qquad (3.9)$$

$$\boxed{-\frac{V^2}{R} - fV - \frac{\partial\phi}{\partial n} = 0}$$

Centrifugal force Coriolis force PGF

- **Geostrophic Flow**

Equation (3.10) implies

$$fV_g = -\frac{\partial \phi}{\partial n} \tag{3.11}$$

where $\boldsymbol{V}$ is replaced by $\boldsymbol{V}_g$ to indicate the geostrophic wind. This implies

$$V_g \propto \delta\phi \text{ if } \delta n \text{ is fixed}$$

and

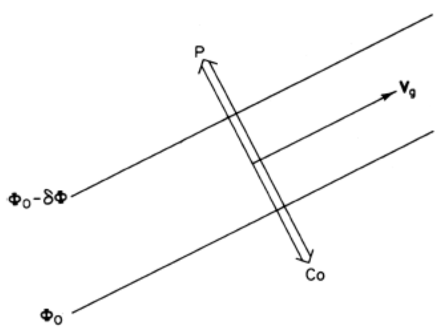
$$V_g \propto \frac{1}{\delta n} \text{ if } \delta\phi \text{ is fixed}$$ (show examples in WX map).

Also, in deriving (3.11), we have assumed

$$\frac{V_g^2}{R} \approx 0,$$

Since V_g is determined by $\partial\phi/\partial n$, we have $R \to \infty$. This means that the local radius of curvature is extremely large.

In other words, the geostrophic flow is straight.



Also note that $\boldsymbol{V}_g$ // height contours (or isobars).

(Because PGF $\perp$ $\boldsymbol{V}_g$ => no pressure gradient along flow direction ($\boldsymbol{t}$) =>$\boldsymbol{V}_g$//height contours)

- **Inertial Flow**

If the geopotential is uniform on an isobaric surface (i.e. pressure gradient is zero or negligible), then there exists no PGF and (3.10)

$$-\frac{V^2}{R} - fV - \frac{\partial \phi}{\partial n} = 0 \qquad (3.10)$$

becomes

$$-\frac{V^2}{R} - fV = 0 \quad \text{or} \quad R = -\frac{V}{f}$$

Centrifugal force   Coriolis force

Flow in this balance is called Inertial Flow.

[Claim] Inertial flow is anticyclonic (clockwise) in northern hemisphere.

Proof: Since *V* is always positive and *f* is positive in northern hemisphere, we have

$$R < 0 .$$

That means the flow is clockwise or anticyclonic (R > 0 if the center of local curvature is located in ***n*** direction).

[Claim] On an *f*-plane, inertial flow follows a circular path.

Proof: Since ϕ is uniform, we have

$$\frac{\partial V}{\partial t} = -\frac{\partial \phi}{\partial s} = 0$$

Thus, V = constant. This implies

$$R = -\frac{V}{f} = \text{constant.}$$

As discussed above, in Northern Hemisphere, $R < 0$, which means the flow is anticyclonic. The period of this anticyclonic oscillation (or inertial flow) can be calculated

$$P = \frac{2\pi|R|}{V} = \frac{2\pi(V/f)}{V} = \frac{2\pi}{f} = \frac{2\pi}{2\Omega\sin\phi} = \frac{2\pi}{2(2\pi/1day)\sin\phi}$$
$$= \frac{1day/2}{\sin\phi} = \frac{half\ day}{\sin\phi} = half\ pendulum\ day$$

[click here for the definition of 1 pendulum day]

- **Cyclostrophic Flow**

For a small-scale circulation, such as a tornado, we have very large Lagrangian Rossby number, such as for a tornado:

$$\frac{V^2/R}{fV} = \frac{V}{fR} \approx \frac{30m/s}{(10^{-4}s^{-1})(200m)} = 1.5x10^3 >> 1.$$

That is, the Lagrangian Rossby number (V/fR) is much greater than 1. Thus, the Coriolis force is negligible and (3.10) can be approximated by the cyclostrophic flow,

$$-\frac{V^2}{R} - \frac{\partial \phi}{\partial n} = 0\ ,$$

Centrifugal force   PGF

which leads to

$$V = \sqrt{-R\frac{\partial \phi}{\partial n}}\ .$$

Since V has to be positive and, of course, real, we require

R and $\dfrac{\partial \phi}{\partial n}$ in opposite signs.

Thus, there are 2 possibilities for the flow circulation.

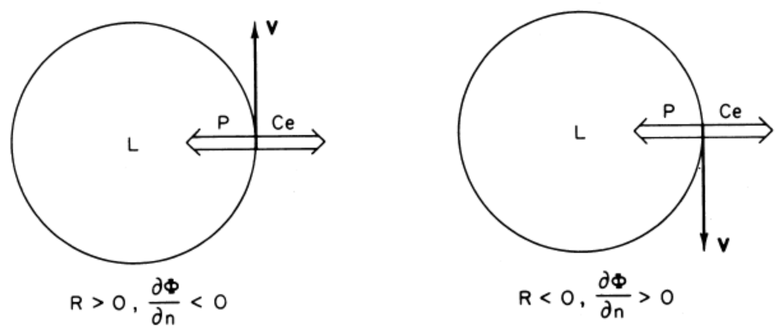


Fig. 3.4: Force balance in cyclostrophic flow: P designates the pressure gradient and Ce the centrifugal force. (Holton and Hakim 2013)

- Source of rotation (vorticity) for tornadic supercell thunderstorms (which tend to produce tornadoes)

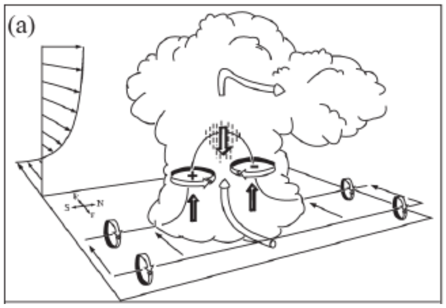


Fig. 8.20: A schematic depicting rotation development and the storm splitting. (a) Rotation development: In the early stage, a pair of vortices forms through tilting of horizontal vorticity associated with the (westerly) environmental shear. (b) Storm splitting: In the later stage, the updraft is split into two convective cells by the upward pressure gradient forces. See text for details. Cylindrical arrows denote the direction of the storm-relative airflow, and heavy solid lines represent vortex lines with the sense of rotation denoted by circular arrows. Shaded arrows represent the forcing promoting new updraft and downdraft acceleration. Vertical dashed lines denote regions of precipitation. Frontal symbols at the surface mark the boundary of cold air outflow. (After Klemp 1987; Reprinted, with permission, from the Annual Review of Fluid Mechanics, Vol. 19 @1987 by Annual Reviews)

- **The Gradient Wind Approximation**

If we assume the flow is inviscid and parallel to height contours (i.e., $\partial\phi/\partial s = 0$), then

$$\frac{DV}{Dt} = 0 .$$

This type of flow is called *gradient flow*. From Eq. (3.10), we have

$$V = -\frac{fR}{2} \pm \sqrt{\left(\frac{fR}{2}\right)^2 - R\frac{\partial\phi}{\partial n}} \tag{3.15}$$

- Requirement for physically possible solutions: V real and positive.
  Let us consider all possibilities of flow configuration

[Possible solutions]			[Root]	[Sign of V]	[Physical]	[Real Situation]
	R	$\frac{\partial \phi}{\partial n}$	$\pm\sqrt{\ }$	(if $\sqrt{\ }$ real)		Yes/No
1a	+	+	+	V<0	No	(Co, PGF, Ce in the same direction)
1b	+	+	-	V<0	No	Same as 1a
2a	+	-	+	V>0	Yes	Regular Low (Fig.3.5a) (baric flow)
2b	+	-	-	V<0	No	
3a	-	+	+	V>0	Yes	Anomalous Low (antibaric) V very large (Fig.3.5c)
3b	-	+	-	V<0	No	
4a	-	-	+	V>0	Yes	Anomalous High V very large (Fig.3.5b)
4b	-	-	-	V>0	Yes	Regular High (baric flow)

- How to sketch a gradient flow and its force balance.

1. Draw a circle.
2. Determine the flow to be cyclonic or anticyclonic, based on R > 0 or R < 0, respectively, and then draw an arrow tangential to the circle to represent the velocity of the flow (***V***).
3. Determine the direction of ***n***, which is always to the left of ***V***.
4. Based on the sign of $\partial \phi / \partial n$ determine the pressure inside the circle to be either a high or a low.
5. Now sketch the PGF (an arrow from High to Low), Ce (centrifugal force, always pointing away from the center of local curvature), and Co (Coriolis force; always to the right of the flow (***V***) in Northern Hemisphere.
6. Now determine the gradient flow is possible to exist in the real atmosphere or not. If it is possible, is it a regular high or low? Is it a baric flow (PGF and Co in opposite directions) or not?

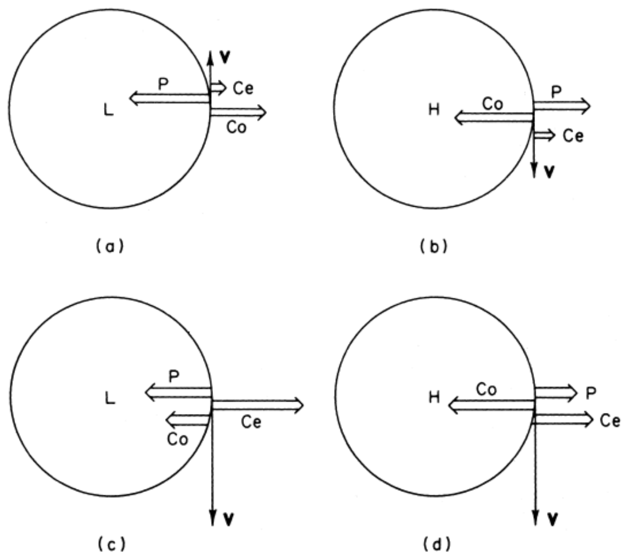
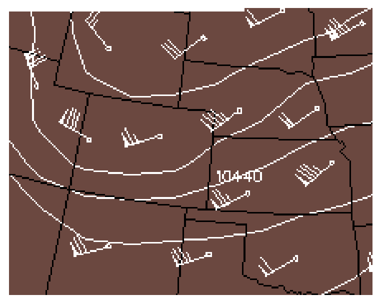


Fig. 3.5 Force balances in the Northern Hemisphere for the four types of gradient flow: (a) regular low (b) regular high (c) anomalous low (d) anomalous high.



Note that flow circulation surrounding hurricanes is approximately in gradient wind balance. Thus, in hurricane prediction models, the hurricanes are often initialized by a bogus vortex which is in gradient wind balance.

- Claim: Near the center of a high is always flat (smaller pressure gradient) and the wind is gentle compared to the region near the center of a low.

Proof:
In order to have a real value of V, for regular and anomalous highs (both R & $\partial\phi/\partial n$ are negative), we require

$$R\frac{\partial\phi}{\partial n} < \left(\frac{fR}{2}\right)^2,$$

since

$$V = -\frac{fR}{2} \pm \sqrt{\left(\frac{fR}{2}\right)^2 - R\frac{\partial\phi}{\partial n}} > 0 \qquad (3.15)$$

This implies

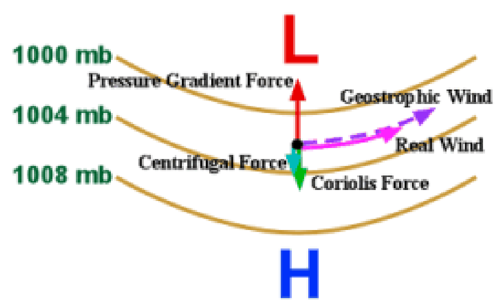
$$|R|\left|\frac{\partial\phi}{\partial n}\right| < \frac{f^2|R|^2}{4} \quad \text{or} \quad \left|\frac{\partial\phi}{\partial n}\right| < \frac{f^2|R|}{4}. \qquad (3.16)$$

Therefore $\partial\phi/\partial n$ should be limited and approaches 0 as R approaches 0.

Thus, near the center of a high is always flat and the wind is gentle compared to the region near the center of a low. Equation (3.15) also implies that an intense low is possible.

- Subgeostrophic flow and supergeostrophic flow

For geostrophic flow, we have



$$-fV_g - \frac{\partial\phi}{\partial n} = 0 \text{ or } \frac{\partial\phi}{\partial n} = -fV_g$$

Substitute it into (3.10) leads to

$$-\frac{V^2}{R} - fV + fV_g = 0 \text{ or } \frac{V_g}{V} = 1 + \frac{V}{fR}.$$

For regular lows: $fR > 0$ and $V > 0$ (always) => $V < V_g$. Flow speed is smaller than the corresponding geostrophic flow. Thus, for regular lows, flow is subgeostrophic flow.

On the other hand, for regular highs, flow is supergeostrophic.