Lecture 15 The Equation of State for Moist Air

(Sec.4.3 of Hess – Equation of State for Moist Air)

A moist air can be treated as a mixture of dry air and water vapor, i.e.

Moist air = dry air + water vapor

We have already learned about how to compute the mean molecular weight of dry air (M_d) from Lecture 2,

$$\bar{M} = \frac{m_1+m_2+m_3+\cdots}{\frac{m_1}{M_1}+\frac{m_2}{M_2}+\frac{m_3}{M_3}+\cdots} = \frac{1}{(m_1/m)/M_1+(m_2/m)/M_2+(m_3/m)/M_3+\cdots}$$

or $\bar{M} = 1/\sum\left(\dfrac{m_i/m}{M_i}\right)$, (2.16)

$M_d = 1/[(m_{N2}/m)/M_{N2}+(m_{O2}/m)/M_{O2}+(m_{Ar}/m)/M_{Ar}]$

$= 1/[0.7551/28+0.2314/32+0.0135/40]$

$= 28.97\ \mathrm{kg\ kmol^{-1}}$

Thus, the mean molecular weight of the moist air ($\bar{M}$) can be computed as follows:

The molecular weights of M_d (dry air) and M_v (water vapor)

$$M_d = 28.97\ kg\ kmol^{-1}, M_v = 18\ kg\ kmol^{-1},$$

and the respective masses of dry air and water vapor of a moist air, m_d and m_v are used to compute the mean molecular weight of moist air

$$\overline{M} = \frac{m_d + m_v}{m_d / M_d + m_v / M_v}.$$

or in terms of the mass ratio m_v/m_d, which is usually referred to as the mixing ratio ($w=m_v/m_d$):

$$\overline{M} = \frac{1+w}{1/M_d + w/M_v}.$$

If we denote M_v/M_d as ε (=18/28.97=0.622), then $\overline{M}$ can be rewritten as

$$\overline{M} = \frac{(1+w)M_d}{1+w/\varepsilon}.$$

We can further compute the gas constant for the moist air,

$$\overline{R} = \frac{R^*}{\overline{M}} = \frac{(1+w/\varepsilon)(R^*/M_d)}{1+w} = \frac{(1+w/\varepsilon)R_d}{1+w}.$$

Thus, the equation of state for a moist air may be written,

$$p\alpha = \overline{R}T = R_d\left(\frac{1+w/\varepsilon}{1+w}\right)T \, .$$

If we define a new temperature variable,

$$T_v = \left(\frac{1+w/\varepsilon}{1+w}\right)T \, , \qquad (5.3.1)$$

which is called virtual temperature, then the equation of state for a moist air may be expressed in a similar way as the equation of state for a dry air

$$p\alpha = R_d T_v \, . \qquad (5.3.2)$$

The physical meaning of the virtual temperature is the temperature that the dry air would have if its pressure and specific volume were equal to those of a given sample of moist air.

Since $\varepsilon = M_v / M_d = 18/28.97 = 0.622$, and practically $w << 1$, Eq. (5.3.1) may be approximated as

$$T_v = \left(\frac{1+w/\varepsilon}{1+w}\right)T \approx (1+w/\varepsilon)(1-w)T \approx$$

$$(1+(1/\varepsilon-1)w - w^2/\varepsilon)T \approx (1+0.61w)T$$

$$T_v = (1+0.61w)T \qquad (5.3.3)$$

Next, we need to express *w* in terms of state variables and known constants, so that it can be measured in the atmosphere.

By definition,

$$w = m_v/m_d = \rho_v/\rho_d \qquad (5.3.4)$$

Using the pressures from the equation of state,

$$p_d = \rho_d (R^*/M_d)T \,,$$

$$e = p_v = \rho_v (R^*/M_v)T \,,$$

to replace the densities, we obtain,

$$w = \left(\frac{M_v}{M_d}\right)\left(\frac{e}{p_d}\right),$$

or

$$w = \frac{M_v}{M_d}\left(\frac{e}{p-e}\right) \approx \frac{M_v}{M_d}\left(\frac{e}{p}\right) = \varepsilon\left(\frac{e}{p}\right) = 0.622\frac{e}{p}. \qquad (5.3.5)$$

Since w is usually less than 0.04, as a result, $T_v - T < 7$ K.

Since the equation of state of dry air is

$$p\alpha = R_d T \;,$$

comparing with (5.3.2), we may conclude that dry air is denser than moist air. This also explains why the moist air rises in the atmosphere.