Dynamical Mountain Meteorology

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Chapter 2. Governing Equations for Mesoscale Motions

2.1 Scales of Atmospheric Flow

- Scaling of atmospheric motions is normally based on observational and theoretical considerations.
- The following horizontal scaling is often used (Orlanski 1975 BAMS):

  - Large (synoptic) scale: L > 2000 km
  - Mesoscale: 2 km < L < 2000 km
    - Meso-α scale: 200 km < L < 2000 km
    - Meso-β scale: 20 km < L < 200 km
    - Meso-γ scale: 2 km < L < 20 km
  - Microscale: L < 2 km

- Sometimes it is more meaningful to adopt a Lagrangian time scale rather than a Eulerian time scale (L/U).

Table 1.2 *Lagrangian time scales and Rossby numbers for typical atmospheric systems. (Adapted after Emanuel and Raymond 1984.)*

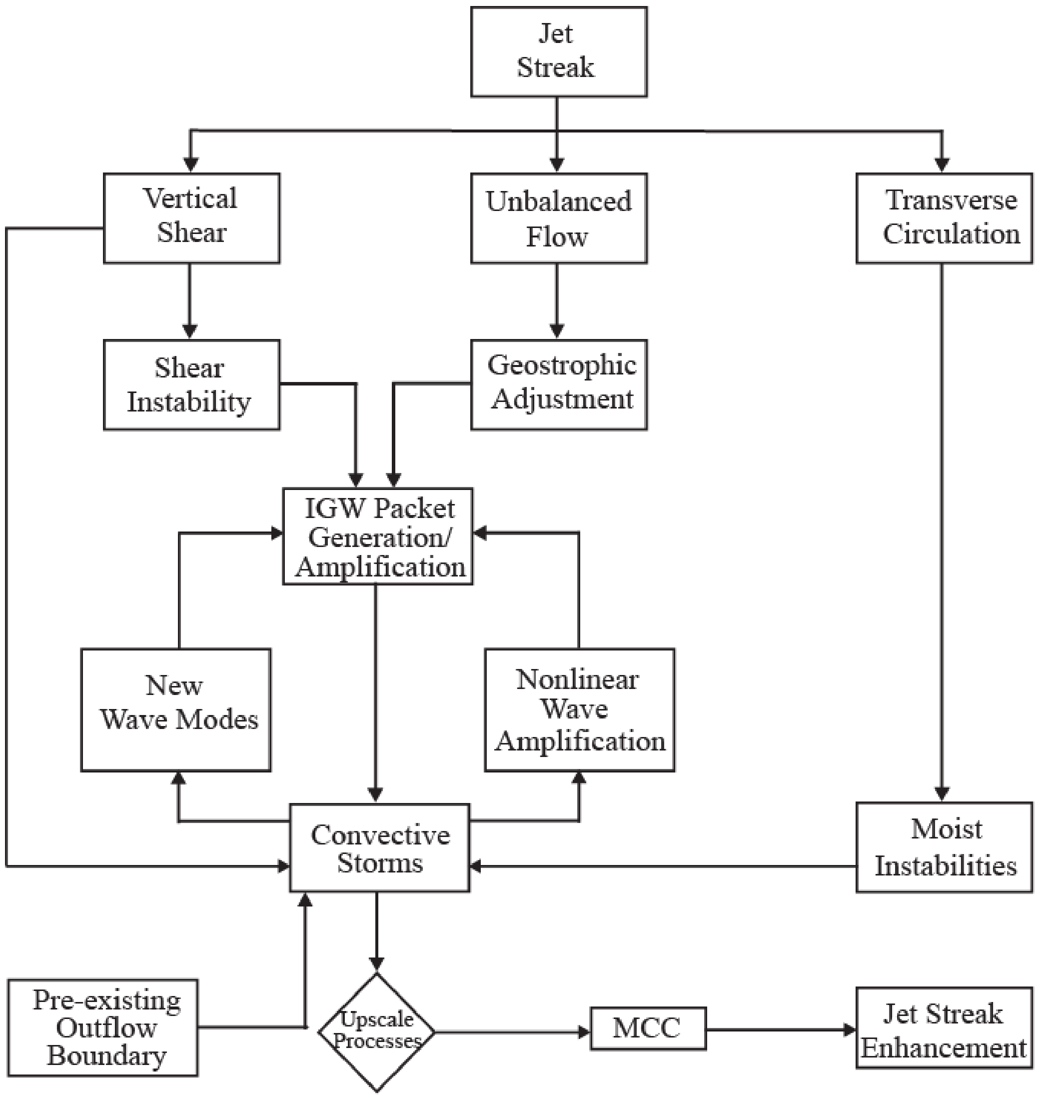
Phenomenon	Time scale	Lagrangian R_o ($\approx \omega/f = 2\pi/fT$)
Tropical cyclone	$2\pi R/V_T$	V_T/fR
Inertia-gravity waves	$2\pi/N$ to $2\pi/f$	N/f to 1
Sea/land breezes	$2\pi/f$	1
Thunderstorms and cumulus clouds	$2\pi/N_w$	N_w/f
Kelvin–Helmholtz waves	$2\pi/N$	N/f
PBL turbulence	$2\pi h/U^*$	U^*/fh
Tornadoes	$2\pi R/V_T$	V_T/fR

where:
R = radius of maximum wind scale, ω = frequency, T = time scale, V_T = maximum tangential wind scale, f = Coriolis parameter, N = buoyancy (Brunt–Vaisala) frequency, N_w = moist buoyancy (Brunt-Vaisala) frequency, U^* = scale for friction velocity, h = scale for the depth of planetary boundary layer.

Table 1: Atmospheric scale definitions (Thunis and Bornstein 1996; Lin 2007)

L_H	Lifetime	Stull (1988)	Pielke (1984)	Orlanski (1975)	Thunis and Bornstein (1996)	Atmospheric Phenomena
10 000km	1 month	Macro	Synoptic Regional	Macro-α	Macro-α	General circulation, long waves
2000 km	1 week			Macro-β	Macro-β	Synoptic cyclones
200 km	1 day			Meso-α	Macro-γ	Fronts, hurricanes, tropical storms, short cyclone waves, mesoscale convective complexes
20 km		Meso	Meso	Meso-β	Meso-β	Mesocyclones, mesohighs, supercells, squall lines, inertia-gravity waves, cloud clusters, low-level jets thunderstorm groups, mountain waves, sea breezes
2 km	1 h	Micro	Micro	Meso-γ	Meso-γ	Thunderstorms, cumulonimbi, clear-air turbulence, heat island, macrobursts
200 m	30 min			Micro-α	Meso-δ	Cumulus, tornadoes, microbursts, hydraulic jumps
20 m	1 min			Micro-β	Micro-β	Plumes, wakes, waterspouts, dust devils
2 m	1 s			Micro-γ	Micro-γ	Turbulence, sound waves
		Micro δ			Micro-δ	

Fig. 1.4: Scale interactions between the jet streak, inertial-gravity waves, and strong convection (Lin 2007, adapted from Koch 1997)



2.2 The Governing Equations of Atmospheric Mesoscale Flow

The governing equations of atmospheric motion and processes are based on:

(a) Newton's 2nd law of motion,
(b) Conservation of mass,
(c) Conservation of energy, and
(d) Ideal gas law.

The Newton's second law of motion is used to derive the horizontal and vertical momentum equations. These momentum equations are also called equations of motion.

The set of momentum equations is called Navier-Stokes equations.

The conservation of mass is used to derive the mass continuity equation or, simply, the continuity equation.

The conservation of energy is then used to derive the first law of thermodynamics and then combined with the ideal gas law to give the thermodynamic energy equation.

The equations governing an atmospheric flow, i.e. the momentum equations, continuity equation, and thermodynamic energy equation, can be derived and expressed in the following form (Lin 2007),

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_{rx}, \tag{2.2.1}$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_{ry}, \tag{2.2.2}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_{rz}, \tag{2.2.3}$$

$$\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0, \tag{2.2.4}$$

$$\frac{D\theta}{Dt} = \frac{\theta}{c_p T} q, \tag{2.2.5}$$

where $D/Dt = \partial/\partial t + u\partial/\partial x + v\partial/\partial y + w\partial/\partial z$ is the *total* or *material derivative*, which represents the change of a certain property within a fluid parcel following the motion, and F_{rx}, F_{ry}, and F_{rz} are the viscous terms in *x*, *y*, and *z* directions, respectively.

- Earth curvature terms need to be considered for larger scale motions, which are often incorporated in NWP models.
- The friction and heat fluxes (sensible heat and latent heat) associated with planetary boundary layer (PBL) processes are normally parameterized by the F_r terms in a numerical model. This proposes a challenging problem in NWP or mesoscale modeling.
- The diabatic heating rate (q) represents the surface heating, elevated latent heating and radiative heating rate per unit mass. Accurate parameterizations of these processes are essential for successful NWP.

- Representations of the diabatic heating in NWP models:
  (a) The surface heating is part of the PBL processes, thus is usually represented by PBL parameterization and land surface parameterization schemes.

(b) The latent heating is represented by cumulus parameterization schemes (subgrid) or microphysical parameterization (grid explicit) schemes.

(c) The radiative processes are represented by radiation (radiative transfer) parameterization schemes.

- The equation set (2.2.1)-(2.2.3) is often referred to as the *Navier-Stokes equations of motion.*

2.3 Approximations of the Governing Equations

- ***Linear Approximation***

Considering an atmosphere on a planetary *f* plane, the momentum equations, continuity equation, and thermodynamic energy equation can be approximated in the following forms, based on perturbation theory:

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_{rx}, \qquad (2.2.1)$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_{ry}, \qquad (2.2.2)$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_{rz}, \qquad (2.2.3)$$

$$\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0, \qquad (2.2.4)$$

$$\frac{D\theta}{Dt} = \frac{\theta}{c_p T}\dot{q}, \qquad (2.2.5)$$

where

$D/Dt = \partial/\partial t + u\partial/\partial x + v\partial/\partial y + w\partial/\partial z$: the total (material) derivative, which represents the rate of change of a certain property within a fluid parcel following the motion,

F_{rx}, F_{ry}, and F_{rz} : the viscous terms or frictional forces per unit mass in the x, y, and z directions, respectively,

c_p : the heat capacity of dry air at constant pressure,

$\dot{q}$: the diabatic heating rate per unit mass in J kg^{-1} s^{-1}.

Q: Is the above set of equations a closed system mathematically?

The system can be closed by adding two additional equations, such as the equation of state for dry air,

$$p = \rho R_d T\ , \tag{2.2.6}$$

and the Poisson's equation

$$\theta = T\left(\frac{p_o}{p}\right)^{R_d/c_p}, \tag{2.2.7}$$

where

θ : the potential temperature,

p_o: a constant reference pressure level (1000 hPa)

R_d : the gas constant for dry air

c_p: heat capacity at constant pressure.

For a moist atmosphere, the temperature in (2.2.6) is replaced by the virtual temperature, which takes into account the moist effects due to latent heat release, and the density is replaced by the total density, which is a sum of the dry air density and the total water density.

The above equation set may be linearized by partitioning the field variables:

$$
\begin{aligned}
&u(t,x,y,z) = U(z) + u'(t,x,y,z)\\
&v(t,x,y,z) = V(z) + v'(t,x,y,z)\\
&w(t,x,y,z) = w'(t,x,y,z)\\
&\rho(t,x,y,z) = \overline{\rho}(x,y,z) + \rho'(t,x,y,z)\\
&p(t,x,y,z) = \overline{p}(x,y,z) + p'(t,x,y,z)\\
&\theta(t,x,y,z) = \overline{\theta}(x,y,z) + \theta'(t,x,y,z)\\
&T(t,x,y,z) = \overline{T}(x,y,z) + T'(t,x,y,z)\\
&\dot{q}(t,x,y,z) = q'(t,x,y,z)
\end{aligned}
\qquad (2.2.8)
$$

where capital letters and overbars represent the basic state and the primes indicate perturbations from the basic state, such as the mesoscale flow fields.

The basic state is assumed to follow Newton's second law of motion, conservation of mass, and the first law of thermodynamics. The basic state is assumed to be in geostrophic balance,

$$U = -\frac{1}{f\overline{\rho}}\frac{\partial \overline{p}}{\partial y} \quad \text{and} \quad V = \frac{1}{f\overline{\rho}}\frac{\partial \overline{p}}{\partial x}, \qquad (2.2.9)$$

and in hydrostatic balance,

$$\frac{\partial \overline{p}}{\partial z} = -\overline{\rho} g, \qquad (2.2.10)$$

where $\bar{p} = \bar{\rho} R_d \bar{T}$. Equations (2.2.9) and (2.2.10) automatically imply the *thermal wind balance* for the basic state

$$U_z = -\frac{g}{f\bar{\theta}}\frac{\partial\bar{\theta}}{\partial y};\ V_z = \frac{g}{f\bar{\theta}}\frac{\partial\bar{\theta}}{\partial x}, \qquad (2.2.11)$$

where $\bar{\theta} = \bar{T}\left(p_o / \bar{p}\right)^{R_d/c_p}$ and subscripts indicate partial differentiations.

The conservation of mass, (2.2.4), may be written as

$$\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0, \qquad (2.2.4)$$

of the basic state leads to

$$\frac{D\bar{\rho}}{Dt} + \bar{\rho}\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) = \frac{1}{c_s^2}\frac{D\bar{p}}{Dt} = \frac{1}{c_s^2}\left(U\frac{\partial\bar{p}}{\partial x} + V\frac{\partial\bar{p}}{\partial y}\right) = 0, \qquad (2.2.12)$$

where c_s is the sound wave speed defined as $\sqrt{\gamma R_d \bar{T}}$.

The last equality of (2.2.12) is consistent with the geostrophic wind relation. Conservation of the basic state thermal energy gives

$$U\frac{\partial\bar{\theta}}{\partial x} + V\frac{\partial\bar{\theta}}{\partial y} = 0, \qquad (2.2.13)$$

which implies no basic state thermal advection by the basic wind, and will be assumed for deriving the perturbation thermodynamic equation.

The left-hand side of (2.2.13) is required to satisfy the constraint that the vertical motion field vanishes at the surface and possibly at the upper boundary for some theoretical studies (Bannon 1986). In the *Eady* (1949) *model* of baroclinic instability, this term is assumed to be 0. In fact, if one assumes $V = 0$, then the above equation is automatically satisfied because $\partial\bar{\theta}/\partial x = (f\bar{\theta}/g)V_z = 0$, based on the basic-state thermal wind relations.

Substituting (2.2.8) with (2.2.9) - (2.2.13) into (2.2.1) - (2.2.5) and neglecting the nonlinear and viscous terms, the perturbation equations for mesoscale motions in the free atmosphere (i.e. above the planetary boundary layer) can be obtained,

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + V\frac{\partial u'}{\partial y} + U_z w' - fv' + \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} = 0 \,, \qquad (2.2.14)$$

$$\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} + V\frac{\partial v'}{\partial y} + V_z w' + fu' + \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial y} = 0 \,, \qquad (2.2.15)$$

$$\frac{\partial w'}{\partial t} + U\frac{\partial w'}{\partial x} + V\frac{\partial w'}{\partial y} - g\frac{\theta'}{\bar{\theta}} + \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z} + \frac{p'}{\bar{\rho}H} = 0 \,, \qquad (2.2.16)$$

$$\frac{1}{c_s^2}\left(\frac{\partial p'}{\partial t} + U\frac{\partial p'}{\partial x} + V\frac{\partial p'}{\partial y}\right) - \frac{\bar{\rho}}{H}w' + \bar{\rho}\nabla\cdot\boldsymbol{V}' = \frac{\bar{\rho}}{c_p\bar{T}}q' \,, \qquad (2.2.17)$$

$$\frac{\partial \theta'}{\partial t} + U\frac{\partial \theta'}{\partial x} + V\frac{\partial \theta'}{\partial y} + \frac{N^2\bar{\theta}}{g}w' = \frac{\bar{\theta}}{c_p\bar{T}}q' \,, \qquad (2.2.18)$$

where N is the Brunt-Vaisala (buoyancy) frequency and H is the scale height, which are defined as

$$N^2 \equiv \frac{g}{\overline{\overline{\theta}}} \frac{\partial \overline{\theta}}{\partial z}, \qquad H \equiv \frac{c_s^2}{g},$$

$$c_s^2 = \gamma R \overline{T}, \quad \text{and} \quad \gamma = \frac{c_p}{c_v}. \tag{2.2.19}$$

Equations (2.2.14) - (2.2.18) form an compressibe fluid system that may include the following types of waves:

(1) pure acoustic waves: c_s finite, $g = 0$, $f = 0$,
(2) acoustic-gravity waves: c_s/g finite, $f = 0$,
(3) pure gravity waves: $c_s \to \infty$, $g \neq 0$, $f = 0$, and
(4) inertia-gravity waves: $c_s \to \infty$, $g \neq 0$, $f \neq 0$.

In general, the system of (2.2.1)-(2.2.7) may include static (buoyant), shear (Kelvin-Helmholtz), symmetric, inertial, and baroclinic instabilities.

➢ *Approximations of the continuity equation*

The continuity equation may be derived to be (see Lin 2007),

$$\frac{\partial \rho'}{\partial t} + \boldsymbol{V} \cdot \nabla \rho' + w \frac{d\overline{\rho}}{dz} + (\overline{\rho} + \rho')\nabla \cdot \boldsymbol{V} = 0, \tag{2.2.20}$$

where $\boldsymbol{V} = (u, v, w)$. For small-amplitude perturbations, the above equation reduces to the following linear form:

$$\frac{1}{\bar{\rho}}\frac{\partial \rho'}{\partial t} + \frac{1}{\bar{\rho}}\bar{\boldsymbol{V}} \cdot \nabla \rho' + \frac{w}{\bar{\rho}}\frac{d\bar{\rho}}{dz} + \nabla \cdot \boldsymbol{V'} = 0 \,. \tag{2.2.21}$$

(a) Fully compressible continuity equation

Note that sound waves are included in the system, thus a special treatment of them are necessary, in order to keep the numerical integrations efficient.

One popular scheme used in NWP models is the *time-splitting scheme* in which smaller time step is adopted for simulating the terms related to sound waves while a larger time step is adopted for simulating the other terms in the governing equations.

Most of the popular NWP models, such as WRF (NCAR), MM5 (PSU-NCAR), ARPS (OU), CSU-RAMS, and COAMPS (NRL) models, adopt the time-splitting scheme.

(b) Anelastic or deep convection continuity equation

$$\nabla \cdot \boldsymbol{V'} - \frac{w'}{H} = 0 \,, \tag{2.3.1}$$

$$\nabla \cdot (\boldsymbol{V'} e^{-z/H}) = 0 \,, \text{ or} \tag{2.3.2}$$

$$\nabla \cdot (\bar{\rho} \boldsymbol{V'}) = 0 \,. \tag{2.3.3}$$

where H is the scale height. The sound waves are filtered out in this approximation.

(c) Incompressible or shallow convection continuity equation

$$\nabla \cdot \boldsymbol{V'} = 0. \tag{2.3.4}$$

This approximation is valid when $L_z / H << 1$. It is important to mention that L_z represents the depth of convection (dry or moist) or disturbance, while H represents the scale height, which is controlled by the basic structure of the atmosphere, instead of the motion.

➢ *Boussinesq Approximation*

A well-known approximation, which has been used widely in theoretical studies, is the *Boussinesq approximation* (Boussinesq 1903; see Spiegel and Veronis 1960), which is equivalent to that: (1) $1/L_z >> 1/H$, (2) density is treated as a constant except where it is coupled to gravity in the buoyancy term of the vertical momentum equation, and (3) replace $\bar{\rho}$ and $\bar{\theta}$ by ρ_o and θ_o, respectively.

➢ For a disturbance with a much larger horizontal scale than vertical scale, the vertical acceleration generally becomes small and may be neglected. This leads to the *linear hydrostatic equation*.

$$\frac{\partial p'}{\partial z} - \left(\frac{g\bar{\rho}}{\bar{\theta}} \right) \theta' = 0. \tag{2.3.12}$$