Dynamical Mountain Meteorology

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Chapter 5 Inertia-Gravity Waves

(Based on Sec. 3.6 of "Mesoscale Dynamics" by Y.-L. Lin)

When the Rossby number ($R_o = U / fL$) becomes smaller, rotational effects need to be considered. In this situation, buoyancy and Coriolis forces can act together as restoring forces and inertia-gravity waves can be generated.

The governing equations are similar to (3.5.1)-(3.5.4), but with three-dimensional and rotational effects included,

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} - fv' + \frac{1}{\rho_o}\frac{\partial p'}{\partial x} = 0 \qquad (3.6.1)$$

$$\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} + fu' + \frac{1}{\rho_o}\frac{\partial p'}{\partial y} = 0 \qquad (3.6.2)$$

$$\frac{\partial w'}{\partial t} + U\frac{\partial w'}{\partial x} - g\frac{\theta'}{\theta_o} + \frac{1}{\rho_o}\frac{\partial p'}{\partial z} = 0 \qquad (3.6.3)$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \qquad (3.6.4)$$

$$\frac{\partial \theta'}{\partial t} + U\frac{\partial \theta'}{\partial x} + \frac{N^2\theta_o}{g}w' = 0. \qquad (3.6.5)$$

The above set of equations can be combined into a single equation for w',

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2\left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2} + \frac{\partial^2 w'}{\partial z^2}\right) + f^2\frac{\partial^2 w'}{\partial z^2} + N^2\left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2}\right) = 0. \qquad (3.6.6)$$

Again, applying the method of normal modes to w' in (x, y, t),

$$w' = \hat{w}(z)\, exp\,[i(kx + ly - \omega t)] \qquad (3.6.7)$$

and substituting it into (3.6.6) lead to

$$\frac{\partial^2 \hat{w}}{\partial z^2} + \frac{K^2(N^2 - \Omega^2)}{\Omega^2 - f^2}\hat{w} = 0, \qquad (3.6.8)$$

where K is the horizontal wave number ($= \sqrt{k^2 + l^2}$).

Similar to the pure gravity wave solutions, the above equation has solutions of the form,

$$\hat{w} = A e^{imz} + B e^{-imz}, \qquad (3.6.9)$$

where *m*, the vertical wave number, is defined as,

$$m^2 = \frac{K^2(N^2 - \Omega^2)}{\Omega^2 - f^2} \qquad (3.6.10)$$

- It is clear from (3.6.9) that wave properties depend on the sign of m^2. Based on the signs of the numerator and denominator in (3.6.10) and on typical values of basic state flow parameters observed for waves in both the atmosphere and ocean (*N* is normally greater than *f*), three different flow regimes may be identified.

  The approximated governing equations and dispersion relations for the different flow regimes are summarized in Table 3.2. Their characteristics are described below.

- (I) The first flow regime occurs when $\Omega^2 > N^2 > f^2$. In this flow regime *m* is imaginary, so term A of (3.6.9) decays exponentially with height, while term B increases exponentially with height.

  $$\hat{w} = A e^{imz} + B e^{-imz}, \qquad (3.6.9)$$

  The flow behavior is similar to the evanescent waves, as discussed in (3.6.10) for the case of $N^2/\Omega^2 < 1$, except that N^2 is required to be greater than f^2, and is referred to as the *high-frequency evanescent flow regime*.

- (Ia) When $\Omega^2 > N^2 >> f^2$, (3.6.10) reduces to

$$m^2 \approx K^2(N^2/\Omega^2 - 1)$$

In this extreme case, Coriolis force plays insignificant roles in the process of wave generation and propagation. Thus the flow becomes the nonrotating (high frequency) evanescent waves, which have been described in Sec. 3.5.

In this flow regime, the governing equation becomes

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2} + \frac{\partial^2 w'}{\partial z^2}\right) + N^2\left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2}\right) = 0. \quad (3.6.6)$$

- (Ib) When $\Omega^2 >> N^2 > f^2$, (3.6.10) reduces to

$$m^2 \approx -K^2. \quad (3.6.11)$$

In this extreme case, both the buoyancy and Coriolis forces play insignificant roles in the process of wave generation and propagation.

The governing equation for the vertical velocity *w'* reduces to what is essentially a three-dimensional version of (3.5.21),

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2} + \frac{\partial^2 w'}{\partial z^2}\right) = 0. \quad (3.6.12)$$

Thus, this extreme flow regime is characterized by *potential* (*irrotational*) *flow*, as discussed in the previous section. In this extreme case, for a steady state flow, the flow regime criterion becomes $L/U << 2\pi/N < 2\pi/f$.

- (II) The second flow regime is when $N^2 > \Omega^2 > f^2$. In this flow regime, m is real and the waves are able to propagate freely in the vertical direction. Thus, the flow regime is referred to as the *vertically propagating inertia-gravity wave regime*.

The two possible mathematical solutions of (3.6.9) represent either an upward or a downward propagation of energy.

$$\hat{w} = Ae^{imz} + Be^{-imz}, \tag{3.6.9}$$

If the wave is generated by a low-level source such as stably stratified flow over a mountain, the radiation condition requires that the wave energy propagate away from the energy source, i.e., upward and away from the orographic forcing.

This also applies to the boundary condition at $z = +\infty$ for elevated thermal forcing. However, both terms in (3.6.9) must be retained in the heating layer (forcing region) and in the layer between the heating base and the lower boundary (the Earth's surface).

$$\hat{w} = Ae^{imz} + Be^{-imz}, \tag{3.6.9}$$

Above the forcing, either orographic forcing or elevated latent heating, term A should be retained to allow the energy to propagate upward, as required by the radiation boundary condition (Section 4.4).

Since the ratio *N/f* is typically large in both the atmosphere and the ocean, this flow regime is applicable to a wide range of intrinsic wave frequencies.

- (IIa) When $N^2 > \Omega^2 >> f^2$ and $O(N) = O(\Omega)$,

$$m^2 = \frac{K^2(N^2 - \Omega^2)}{\Omega^2 - f^2} \tag{3.6.10}$$

The above equation, (3.6.10), reduces to

$$m^2 \approx K^2\left(\frac{N^2}{\Omega^2} - 1\right). \tag{3.6.13}$$

In this limit, rotational effects may be ignored, and the flow belongs to the *nonrotating* or *pure gravity wave regime*, as described in Section 3.5.

Notice that for this extreme case, the flow regime criterion become

$2\pi / N < L/U << 2\pi / f$ or $U/NL < 1/2\pi << R_o$ for a steady state flow. This implies that in order to generate pure gravity waves, the Rossby number of the basic state flow must be very large, normally much greater than 1.

The governing equation for the vertical velocity w' in this extreme case becomes

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2} + \frac{\partial^2 w'}{\partial z^2}\right) + N^2 \left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2}\right) = 0 \,. \qquad (3.6.14)$$

- (IIb) When $N^2 >> \Omega^2 >> f^2$, (3.6.10) reduces to

$$m^2 \approx \left(\frac{KN}{\Omega}\right)^2 . \qquad (3.6.15)$$

This is identical to the *nonrotating hydrostatic gravity wave regime*, as discussed earlier.

The governing equation for w' in this extreme case ($N^2 >> \Omega^2 >> f^2$) becomes

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \frac{\partial^2 w'}{\partial z^2} + N^2 \left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2}\right) = 0 \,. \qquad (3.6.16)$$

The flow regime criterion becomes $2\pi / N << L/U << 2\pi / f$ for a steady state flow.

This implies that the fluid parcel advection time is much longer than the period of buoyancy oscillation allowing the disturbance to propagate vertically, but much shorter than the inertial oscillation period. *In this nonrotating hydrostatic wave regime, only vertically propagating waves are allowed.*

- (IIc) When $N^2 >> \Omega^2 > f^2$ and $O(\Omega) = O(f)$, (3.6.10) reduces to

$$m^2 \approx \frac{K^2 N^2}{\Omega^2 - f^2} \qquad (3.6.17)$$

The flow response belongs to the *hydrostatic inertia-gravity wave regime*. Thus, the governing equation for w' becomes

$$\left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 + f^2\right]\frac{\partial^2 w'}{\partial z^2} + N^2\left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2}\right) = 0. \qquad (3.6.18)$$

For a basic flow with $N = 0.01\ \mathrm{s}^{-1}$ and $f = 10^{-4}\mathrm{s}^{-1}$, the horizontal scale of typical hydrostatic inertia-gravity waves is on the order of 100 km.

- (III) The third flow regime is when $N^2 > f^2 > \Omega^2$. In this flow regime, *m* is imaginary. Similar to the first flow regime ($\Omega^2 > N^2 > f^2$), disturbances decay exponentially in the vertical away from the wave energy source. However, the wave frequency is low, thus the flow response is referred to as the *low-frequency evanescent flow regime*.

- (IIIa) When $N^2 > f^2 >> \Omega^2$, inertial accelerations play an insignificant role in wave generation and propagation. The flow response is similar to a *quasi-geostrophic flow*. In this limiting case, (3.6.10) reduces to

$$m^2 \approx \frac{-K^2 N^2}{f^2}. \qquad (3.6.19)$$

In this case, the fluid motion is quasi-horizontal and the governing equation for the vertical velocity w' becomes

$$f^2\frac{\partial^2 w'}{\partial z^2} + N^2\left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2}\right) = 0. \qquad (3.6.20)$$

The horizontal scale for this type of *quasi-geostrophic flow* is on the order of 1000 km for typical values of $N = 0.01\ \mathrm{s}^{-1}$ and $f = 10^{-4}\ \mathrm{s}^{-1}$. The Rossby number of the basic state flow in this case is much smaller than 1.

In order to better understand the basic wave dynamics, we consider the case of hydrostatic inertia-gravity waves in a quiescent fluid ($U = 0$).

Substituting $\varphi' = \tilde{\varphi}\exp(kx + ly + mz - \omega t)$, where $\varphi = u, v, w, p,$ or θ, into (3.6.1) - (3.6.5) and using the hydrostatic form of (3.6.3) lead to the following *polarization relationships* in wave number space,

$$\tilde{u} = \frac{1}{\rho_o}\left(\frac{\omega k + ilf}{\omega^2 - f^2}\right)\tilde{p}\,;\ \tilde{v} = \frac{1}{\rho_o}\left(\frac{\omega l - ikf}{\omega^2 - f^2}\right)\tilde{p}\,;\ \tilde{w} = \frac{-\omega}{\rho_o m}\left(\frac{k^2 + l^2}{\omega^2 - f^2}\right)\tilde{p};\ \tilde{\theta} = \frac{iN^2\theta_o}{\rho_o g m}\left(\frac{k^2 + l^2}{\omega^2 - f^2}\right)\tilde{p}. \quad (3.6.21)$$

The above relationships can be depicted in Fig. 3.10.

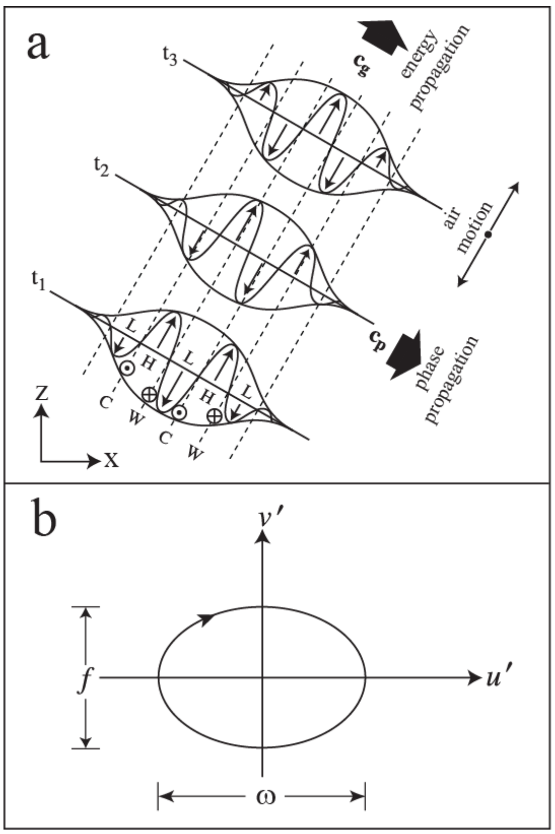

Fig. 3.10: (a) Similar to Fig. 3.9, except for a hydrostatic inertia-gravity wave with $m < 0$, $k > 0$, $l = 0$, $\omega > 0$, and $f > 0$. Meridional (i.e. north-south) perturbation wind velocities (v') are shown by arrows pointed into and out of the page. (b) The projection of fluid particle motion associated with a hydrostatic inertia-gravity wave onto the horizontal plane is an ellipse with ω/f as the ratio of major and minor axes. The velocity vector associated with a plane inertia-gravity wave rotates anticyclonically in the Northern Hemisphere with height for upward energy propagation. (Adapted after Hooke 1986)

In the special case of two-dimensional flow ($\partial/\partial y = 0$ or l = 0), it can be shown that the following solutions satisfy the two-dimensional form of (3.6.6),

$$u' = \tilde{u}\cos(kx + mz - \omega t)$$
$$v' = \left(\frac{f}{\omega}\right)\tilde{u}\sin(kx + mz - \omega t) . \qquad (3.6.22)$$

It can be easily shown from the above equations that the velocity vector associated with a plane inertia-gravity wave rotates anticyclonically with time in the Northern Hemisphere.

The projection of the motion on the horizontal plane is an ellipse where ω/f is the ratio between the major and minor axes, as depicted in Fig. 3.10b.

The velocity vector associated with an inertia-gravity wave rotates anticylonically with height for upward energy propagation. The particle motion and phase relationship for a Poincaré wave propagating on the ocean surface is similar to that of a hydrostatic inertia-gravity wave.

It can also be shown that the ratio of the vertical to horizontal components of the group velocity vector for a two-dimensional, hydrostatic inertia-gravity wave is given by

$$\left|c_{gz}/c_{gx}\right| = \left|k/m\right| = \sqrt{\omega^2 - f^2}/N . \qquad (3.6.23)$$

Therefore, the wave energy of a hydrostatic inertia-gravity wave propagates more horizontally than that of a pure gravity wave with the same wave frequency.

Occasionally, it is observed that large-scale pressure gradients over the ocean are considerably smaller than those over the continents, which leads to a balance between Coriolis and centrifugal forces.

Under this situation, fluid parcels follow circular paths, rotating in an anticyclonic sense in the horizontal plane, and that have an oscillation period of one pendulum day ($2\pi / f$). This type of flow is called *inertial flow or inertial oscillation*.

The radius (R) of curvature of the oscillation can be shown to be $R = -V / f$, where V is the non-negative horizontal wind speed along the direction tangential to the local velocity in the natural coordinates. The negative sign of R indicates the oscillation is anticyclonic (clockwise). The inertial oscillation has been used to explain the formation of low-level jets over the Great Plains to the east of the US Rocky Mountains (Sec. 10.6).

It can be shown that for this particular example

$$\boldsymbol{c_g} \cdot \boldsymbol{k} = 0 \,. \qquad (3.6.24)$$

This indicates that the group velocity vector for inertia-gravity waves is perpendicular to both the wave number vector and the phase velocity vector.